COMPLEX ALGEBRAIC GEOMETRY AND CALCULATION OF MULTIPLICITIES FOR INDUCED REPRESENTATIONS OF NILPOTENT LIE GROUPS

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ABSTRACT. Let G be a connected, simply connected nilpotent Lie group, H a Lie subgroup, and σ an irreducible unitary representation of H. In a previous paper, the authors and G. Grelaud gave an explicit direct integral decomposition (with multiplicities) of $\operatorname{Ind}(H \uparrow G, \sigma)$. One consequence of that work was that the multiplicity function was either a.e. infinite or a.e. bounded. In this paper, it is proved that if the multiplicity function is bounded, its parity is a.e. constant. The proof is algebraic-geometric in nature and amounts to an extension of the familiar fact that for almost all polynomials over R of fixed degree, the parity of the number of roots is a.e. constant. One consequence of the methods is that if G is a *complex* nilpotent Lie group and H a complex Lie subgroup, then the multiplicity is a.e. constant.

1. Let K be a connected, closed subgroup of the connected, simply connected real nilpotent Lie group G, and let χ be an irreducible unitary representation of K. In [2], the authors and G. Grelaud gave a direct integral decomposition of the induced representation $\rho = \operatorname{Ind}(K \uparrow G, \chi)$ into irreducibles: $\rho \cong \int_{G^{\widehat{}}}^{\oplus} m(\pi) \cdot \pi \, d\nu(\pi)$. The spectrum, $\operatorname{supp}\nu$, and the multiplicities, $m(\pi)$, were computed in a natural geometric way in terms of coadjoint orbits. We were also able to show, as a consequence of our methods, that

$$\text{either } m(\pi) \equiv \infty \text{ or there is a bound } N \text{ with } m(\pi) \leq N$$

(of course, this holds ν-almost everywhere). In the examples computed in [2], it was also always true that when $m(\pi)$ was finite, it had constant parity (was always even or always odd) ν-a.e. Roger Howe asked us whether this was always the case, and one of the main purposes of our paper is to prove that it is. Our methods also enable us to get a multiplicity result for complex nilpotent Lie groups as well. We feel that this approach may prove useful in other areas, notably the decomposition of $\pi|_K$ for $\pi \in G^{\widehat{}}$ (a question currently under investigation).

The basic results we prove in this paper are these:

THEOREM 1. *Let $\mathfrak{g}$ be a complex nilpotent Lie algebra and $\mathfrak{k}$ a complex subalgebra, and let G, K be the associated simply connected Lie groups (regarded as real Lie groups). Let χ be an irreducible unitary representation of K, and let*

$\rho = \text{Ind}(K \uparrow G, \chi)$. *Then* ρ *has uniform multiplicity: in the direct integral decomposition of* ρ *into irreducibles,*

$$\rho \cong \int_{G^{\hat{}}}^{\oplus} m(\pi) \cdot \pi \, \nu(d\pi),$$

we have $m(\pi) = constant$, ν-*a.e.* (*The constant may be* ∞.)

THEOREM 2. *Let* G_0 *be a real connected, simply connected nilpotent Lie group,* K_0 *a closed connected subgroup, and* χ *an irreducible unitary representation of* K_0. *Set* $\rho = \text{Ind}(K_0 \uparrow G_0, \chi)$. *Then in the direct integral decomposition of* ρ *into irreducibles,* $\rho \cong \int_{G_0^{\hat{}}}^{\oplus} m(\pi) \cdot \pi \, \nu(d\pi)$, *either*

   (i) $m(\pi) \equiv +\infty$ *or*
   (ii) $m(\pi)$ *is bounded and has constant parity for* ν-*a.e.* π.

The ideas behind the two proofs are similar; both use complex algebraic geometry. However, the details are quite different. In Theorem 1, we are working directly with varieties over **C**, while in Theorem 2 we need to complexify the Kirillov orbit picture. Thus the two proofs are virtually independent. In both cases, we must outline the detailed concordance between the real and complex pictures for the main facts of [**2**], which were proved there only for real fields, and for the Pukanszky parametrization of all coadjoint orbits. Regarding [**2**], we shall try to state clearly the results and adaptations needed for this paper, in order to make this account self-contained.

The following is the organization of the rest of this paper: §2 contains an account of the main results of [**2**], and concludes with a sketch of the proofs of Theorems 1 and 2. In §3, we give the main results we need from algebraic geometry. §4 gives some results on coadjoint orbits, needed for the proof of Theorem 1; we prove Theorem 1 in §5. In §6, we give results comparing coadjoint orbits for a nilpotent Lie group and its complexification; we use these in §7, where Theorem 2 is proved. §8 contains some examples.

It is a pleasure to thank David Rohrlich for his assistance with some of the results from algebraic geometry cited in this paper.

2. We shall repeatedly refer to Pukanszky's method [**6**] for parametrizing all the orbits in a vector space under the action of a connected unipotent Lie group; we shall also need a number of basic facts from the paper [**2**]. Here we assemble these results for future reference.

In what follows, we use **F** to stand for **R** or **C**; in fact, for most statements **F** could be any field of characteristic 0. The theorem in [**6**] (see also [**1**]) on parametrization of orbits is given for **F** = **R**, but the proof applies *verbatim* when **F** = **C**.

If V is a vector space over **F** and $\mathfrak{g}$ is a Lie algebra (over **F**) acting nilpotently on V, we may assume that $\mathfrak{g}$ is nilpotent by factoring out the kernel. Then $G = \{e^X : X \in \mathfrak{g}\}$ is a simply connected nilpotent Lie group (over **F**) acting unipotently on V. Let $\{l_1, \ldots, l_n\}$ be a Jordan-Hölder basis for V. Then $V_j = \mathbf{F}$-span$\{l_{j+1}, \ldots, l_n\}$ is G-invariant for all j ($V_n = (0)$ by definition), and we obtain quotient actions of $\mathfrak{g}$ and G on V/V_j which commute with the canonical projections $P_j : V \to V/V_j$. If $l \in V$, define

$$e_j(l) = \dim_{\mathbf{F}}(G \cdot P_j(l)), \qquad e(l) = (e_1(l), \ldots, e_n(l)) \in \mathbf{Z}^n.$$

The stabilizer $\text{Stab}_G(l) = \{x \in G : x \cdot l = l\}$ is a connected Lie subgroup of G whose Lie algebra is $\mathfrak{p}(l) = \{X \in \mathfrak{g} : X \cdot l = 0\}; \mathfrak{p}(l)$ is an algebra over $\mathbf{F}$, and the orbit $G \cdot l$ is a complex manifold if $\mathbf{F} = \mathbf{C}$. Similar remarks hold for the quotient actions.

The result from [6] (as modified in [1]) that we need is

(2.1) THEOREM (PARAMETRIZATION OF ORBITS). *Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbf{F}$ acting nilpotently on a $\mathbf{F}$-vector space V. Fix a Jordan-Hölder basis for V; define G, the V_j, the quotient actions $G \times V/V_j \to V/V_j$, and the n-tuples $e(l)$ for $l \in V$ as above. Let $\mathcal{E} = \{e: \text{there exists } l \in V \text{ with } e_j = e_j(l) \text{ for } 1 \le j \le n\};$ for $e \in \mathcal{E}$, define $U_e = \{l \in V : e(l) = e\}$. Then the U_e partition V, and*

(i) *$e \in \mathcal{E} \Rightarrow e_j = e_{j-1}$ or $e_j = e_{j-1} + 1$, $\forall j$ (we set $e_0 = 0$). In particular, $\mathcal{E}$ is finite.*

(ii) *There is an ordering of indices $\mathcal{E} = \{e^{(1)} > \cdots > e^{(r)}\}$ such that ($\forall e \in \mathcal{E}$) $\bigcup_{e' \ge e} U_{e'}$ is a Zariski-open set in V. In particular, every U_e is Zariski-open in the Zariski-closed set $V \sim \bigcup_{e' > e} U_{e'}$; furthermore, $U_{e^{(1)}}$ is Zariski-open in V.*

(iii) *Each layer U_e is G-invariant, and all orbits in U_e have the same $\mathbf{F}$-dimension.*

Fix $e \in \mathcal{E}$, and define index sets $S(e) = \{j : e_j = 1 + e_{j-1}\}, T(e) = \{j : e_j = e_{j-1}\}$. There is a direct sum decomposition $V = V_{T(e)} \oplus V_{S(e)}$, where $V_{T(e)} = \mathbf{F}\text{-span}\{l_j : j \in T(e)\}, V_{S(e)} = \mathbf{F}\text{-span}\{l_j : j \in S(e)\}$. Then

(iv) *$\Sigma_e = U_e \cap V_{T(e)}$ is a nonempty set in $V_{T(e)}$, and is Zariski-open in the Zariski-closed set $V_{T(e)} \sim (\bigcup_{e' > e} U_{e'})$.*

(v) *The elements of Σ_e give a cross-section for the G-orbits in U_e, and the semialgebraic set $\Sigma = \bigcup_e \Sigma_e$ is a cross-section for all G-orbits in V.*

(vi) *There is an $\mathbf{F}$-birational nonsingular map $Q_e: \Sigma_e \times V_{S(e)} \to U_e$ such that for each $l \in \Sigma_e$, Q_e maps $\{l\} \times V_{S(e)}$ to $G \cdot l$. This map is polynomial on $V_{S(e)}$ for each $l \in \Sigma_e$, and the inverse $P_e: U_e \to \Sigma_e \times V_{S(e)}$ is given by*

$$P_e(l) = (G \cdot l \cap V_{T(e)}, \ \pi_S(l)),$$

where π_S, π_T are the projections identifying V with $V_{S(e)} \oplus V_{T(e)}$. In particular, $\pi_S \circ Q_e: \Sigma_e \times V_{S(e)} \to V_{S(e)}$ is the projection on the second factor. (This shows that each orbit is the graph of a polynomial.)

The sets U_e are called "layers" in V, and the collection $\{U_e\}$ is an "$\mathcal{E}$-layering" of V. We write $\tilde{P}_e = \pi_T \circ P_e: U_e \to \Sigma_e$.

In Kirillov theory, one applies this result to the coadjoint action: a real nilpotent Lie group G_0 acts on its Lie algebra $\mathfrak{g}_0$ by Ad, and thus on $\mathfrak{g}_0^* = \text{Hom}_{\mathbf{R}}(\mathfrak{g}_0, \mathbf{R})$ by the coadjoint action Ad^*. If $X_1, \ldots, X_n$ is a strong Malcev basis for $\mathfrak{g}_0$ (i.e., $\mathbf{R}\text{-span}\{X_1, \ldots, X_j\}$ is an ideal, $1 \le j \le n$), the dual basis $\{l_1, \ldots, l_n\}$ in $\mathfrak{g}_0^*$ is Jordan-Hölder and determines an $\mathcal{E}$-layering of $\mathfrak{g}_0^*$.

In this paper we must consider some other situations. Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbf{C}$, and let $\mathfrak{g}_{\mathbf{R}}$ be $\mathfrak{g}$ with the field reduced to $\mathbf{R}$. Then $G = \exp \mathfrak{g}$ is a complex Lie group; it acts on both $\mathfrak{g}$ and $\mathfrak{g}_{\mathbf{R}}$. Thus we have two coadjoint actions of G on $\mathfrak{g}^* = \text{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathbf{C})$, and on $\mathfrak{g}_{\mathbf{R}}^* = \text{Hom}_{\mathbf{R}}(\mathfrak{g}_{\mathbf{R}}, \mathbf{R})$. Theorem 2.1 applies to both of them. If one chooses nicely related strong Malcev bases in $\mathfrak{g}$ and $\mathfrak{g}_{\mathbf{R}}$, then the dual bases in $\mathfrak{g}^*, \mathfrak{g}_{\mathbf{R}}^*$ will be related in a way that lets us set up a concordance between the layerings in $\mathfrak{g}^*$ and $\mathfrak{g}_{\mathbf{R}}^*$. This will be done explicitly in §4, as the first step in proving Theorem 1. In proving Theorem 2, we face a different problem:

we start with a real algebra $\mathfrak{g}_0$ and form its complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbf{C}$, and we must relate layerings in $\mathfrak{g}_0^*$ with those in $\mathfrak{g}^* = \operatorname{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathbf{C})$. Here we have different groups G_0, G (G is the group corresponding to $\mathfrak{g}$) acting on different spaces $\mathfrak{g}_0^*, \mathfrak{g}^*$. We analyze these actions in §6.

The basic result of [2] is stated in terms of the $\mathcal{E}$-layering. We recall it here because we will have to devise some variants. Let $\mathfrak{k}_0$ be a (real) subalgebra of the real nilpotent Lie algebra $\mathfrak{g}_0$; let $G_0 = \exp \mathfrak{g}_0$, $K_0 = \exp \mathfrak{k}_0$. Let $P \colon \mathfrak{g}_0^* \to \mathfrak{k}_0^*$ be the canonical projection, and let $\chi \in \widehat{K_0}$ be an irreducible unitary representation with coadjoint orbit $\mathcal{O}_\chi = K_0 \cdot f_0 \subseteq \mathfrak{k}_0^*$ (here, $f_0 \in \mathfrak{k}_0^*$). Fix a strong Malcev basis $X_1, \ldots, X_n$ in $\mathfrak{g}_0$, let $l_1, \ldots, l_n$ be the dual basis in $\mathfrak{g}_0^*$, and partition $\mathfrak{g}_0^*$ into layers $\{U_e \colon e \in \mathcal{E}\}$. Now let $e \in \mathcal{E}$ be the largest index such that $U_e \cap P^{-1}(\mathcal{O}_\chi) \neq \varnothing$; this intersection is Zariski-open and dense in the (irreducible) algebraic variety $P^{-1}(\mathcal{O}_\chi)$, and $\Sigma_e^\chi = \pi_T \circ P_e(U_e \cap P^{-1}(\mathcal{O}_\chi))$ is a semialgebraic set of representatives for the (generic) G_0-orbits in $\mathfrak{g}_0^*$ that meet $P^{-1}(\mathcal{O}_\chi)$.

(2.2) THEOREM [2] (MULTIPLICITY THEOREM). *Let $\mathfrak{g}_0$ be a real nilpotent Lie algebra, $\mathfrak{k}_0$ a subalgebra; let G_0, K_0 be the corresponding connected, simply connected groups, and let $\chi \in \widehat{K_0}$. Defining an $\mathcal{E}$-layering for $\mathfrak{g}_0^*$ as above, let $[\nu]$ be the canonical measure class on Σ_e^χ, and write $G_0 \cdot l = \operatorname{Ad}^*(G_0)l$ for $l \in \mathfrak{g}_0^*$. Define the "defect index"*

$$\tau_0 = \operatorname{generic}(\dim_{\mathbf{R}} G_0 \cdot l \colon l \in P^{-1}(\mathcal{O}_\chi))$$
$$- 2\operatorname{generic}(\dim_{\mathbf{R}} K_0 \cdot l \colon l \in P^{-1}(\mathcal{O}_\chi)) + \dim_{\mathbf{R}} \mathcal{O}_\chi.$$

Then if $\tau_0 > 0$, we have

$$\rho = \operatorname{Ind}(K_0 \uparrow G_0, \chi) \cong \int_{\Sigma_e^\chi} \infty \cdot \pi_l \, d\nu(l).$$

If $\tau_0 = 0$, then for ν-a.e. $l \in \Sigma_e^\chi$, we have
   (i) *$G_0 \cdot l \cap P^{-1}(\mathcal{O}_\chi)$ is a closed submanifold.*
   (ii) *There is a uniform bound for*

(2.3) $m(l) = $ *number of connected components in $G_0 \cdot l \cap P^{-1}(\mathcal{O}_\chi)$.*

   (iii) *$m(l) = $ number of $\operatorname{Ad}^*(K_0)$-orbits in $G_0 \cdot l \cap P^{-1}(\mathcal{O}_\chi)$, and*

$$\rho \cong \int_{\Sigma_e^\chi}^{\oplus} m(l) \pi_l \, d\nu(l).$$

Note. We will recall the definitions of semialgebraic sets and their canonical measure classes in the next section.

Since χ is monomial, it will always be possible for us to assume, by changing K_0, that χ is 1-dimensional. (In the case of Theorem 1, we will have to show that every irreducible representation of a complex group is induced from a 1-dimensional representation of a complex subgroup—see §4.) Thus we shall assume from now on that $\dim \chi = 1$. Then

(2.4) $$\chi(y) = e^{2\pi i f_0(\log y)}$$

where $f_0 \in \mathfrak{k}_0^*$ is a Lie homomorphism. Thus $\mathcal{O}_\chi = K_0 \cdot f_0$ is a single point and $P^{-1}(\mathcal{O}_\chi)$ is flat. In proving Theorems 1 and 2, we will make use of a subsidiary result used in [2] for proving Theorem 2.2.

(2.5) PROPOSITION. *With notation as in Theorem 2.2, assume that* $\dim \chi = 1$; *define* f_0 *as in* (2.4), *and let* $f' \in P^{-1}(f_0)$. *Then there is a subspace* E_1^0 *of* $\mathfrak{k}_0^\perp$ *such that*

(2.6)
$$\rho \cong \int_{f'+E_1^0}^{\oplus} \pi_l \, d\mu(l),$$

where $d\mu(l)$ *is Lebesgue measure.*

We can now explain the idea behind the proof of Theorem 2. The formula (2.6) makes it plausible that the number $m(l)$ of Theorem 2.2 is equal (a.e.) to $\mathrm{Card}(G \cdot l) \cap (f' + E_1^0)$; we shall prove this (modulo results in [2]) in §3. (We may assume that $\tau_0 = 0$, since otherwise $m(l) \equiv \infty$.) We thus wish to know that $\mathrm{Card}(G \cdot l) \cap (f' + E_1^0)$ has constant parity. To prove this, complexify everything: let $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$, $\mathfrak{k} = \mathfrak{k}_0 \otimes_{\mathbf{R}} \mathbf{C}$, and let G, K be the corresponding Lie groups. Similarly, let $\mathfrak{g}^* = \mathfrak{g}_0^* \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{g}_0^* + i\mathfrak{g}_0^*$, $\mathfrak{k}^* = \mathfrak{k}_0^* \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{k}_0^* + i\mathfrak{k}_0^*$. In $\mathfrak{g}^*$ we consider $f' + \mathbf{C}E_1^0$, the Zariski-closure of $f' + E_1^0 \subseteq \mathfrak{g}_0^*$. Generically, orbits $G \cdot l$ for the action of G on $l \in \mathfrak{g}^*$ must meet $f' + \mathbf{C}E_1^0$ in isolated points, just as orbits $G_0 \cdot l$ meet $f' + E_1^0$ discretely if $l \in \mathfrak{g}_0^*$. Let $(\Sigma_e^\chi)^{\mathbf{C}}$ be the Zariski-closure in $\mathfrak{g}^*$ of $\Sigma_e^\chi \subseteq \mathfrak{g}_0^*$. The Pukanszky parametrization of orbits applies to both the action of G on $\mathfrak{g}^*$ and the action of G_0 on $\mathfrak{g}_0^*$, and there is a natural concordance between the two situations. In the complex situation we obtain a map taking a point $l \in f' + \mathbf{C}E_1^0$ to its orbit representative, which lies in $(\Sigma_e^\chi)^{\mathbf{C}}$. This map is rational and nonsingular, and it is finite-to-one on a Zariski-open set. With a little effort it can be regarded as a polynomial map from $f' + \mathbf{C}E_1^0$ to $(\Sigma_e^\chi)^{\mathbf{C}}$. It can be shown that both varieties are irreducible and have the same dimension over $\mathbf{C}$. By a result in algebraic geometry, we can remove negligible sets from each variety and obtain a covering map (in the Euclidean topology). Thus

$$n(l) = \mathrm{Card}[G \cdot l \cap (f' + \mathbf{C}E_1^0)]$$

is defined and constant on a topologically connected Zariski-open set in $(\Sigma_e^\chi)^{\mathbf{C}}$.

Now suppose that l is real: $l \in (\Sigma_e^\chi)^{\mathbf{C}} \cap (\mathfrak{g}_0^* + i0)$. The nonreal points in $G \cdot l \cap (f' + \mathbf{C}E_1^0)$ come in conjugate pairs, and hence the parity of $n_0(l) = \mathrm{Card}[G \cdot l \cap (f' + E_1^0)]$ is constant on a Zariski-open set in Σ_e^χ. It is not hard to show that for real l, $G \cdot l \cap (\mathfrak{g}_0^*) = G_0 \cdot l$. Thus $n_0(l)$ has essentially constant parity. Finally, one observes that $n_0(l) = m(l)$—except for a few measure-theoretic details, this is Proposition 2.5—to complete the proof.

Theorem 1 is similar but easier. The main observation is that in this setting, $\mathfrak{g}_{\mathbf{R}}^*$ and $\mathfrak{k}_{\mathbf{R}}^*$ can be regarded as complex vector spaces, and E_1 is a complex subspace. Thus the map of $f' + \mathbf{C}E_1$ to Σ_e^χ obtained by restricting $\tilde{P}$ is essentially a covering map, and hence $n(l) = \mathrm{Card}(G \cdot l \cap (f' + E_1))$ is essentially constant. One then shows (via Proposition 2.5) that $n(l) = m(l)$ to complete the proof.

3. Here we give some facts from algebraic geometry that will be used repeatedly. In a few cases, we give proofs for facts that have reasonably direct analytic proofs; these results are probably familiar to geometers. We shall need a few results from algebraic geometry besides those cited in this section; they will be given as needed.

We use $\mathbf{F}$ to refer to either $\mathbf{R}$ or $\mathbf{C}$. We generally regard $\mathbf{R}^n$ as a subset of $\mathbf{C}^n$ (in the obvious way); thus we speak of the (complex) Zariski closure of a variety

in $\mathbf{R}^n$. We generally use a subscript "0" to refer to subsets of $\mathbf{R}^n$. If X is any subset of $\mathbf{C}^n$, its (complex) Zariski closure is denoted by $X^{\mathbf{C}}$; if $X_0 \subset \mathbf{R}^n$, its Zariski closure in $\mathbf{R}^n$ is denoted by $X_0^{\mathbf{R}}$. When we are dealing with $\mathbf{F}$, we denote the Zariski closure of a set S by $S^{\#}$.

(3.1) LEMMA. *For any* $X_0 \subset \mathbf{R}^n$,
(i) $(X_0^{\mathbf{R}})^{\mathbf{C}} = X_0^{\mathbf{C}}$;
(ii) $X_0^{\mathbf{C}} \cap \mathbf{R}^n = X_0^{\mathbf{R}}$;
(iii) $X_0^{\mathbf{R}}$ *irreducible in* $\mathbf{R}^n \Rightarrow X_0^{\mathbf{C}}$ *irreducible in* $\mathbf{C}^n$.

(Recall that X is irreducible if it is not the union of two proper subvarieties.) These are all easy, so we omit the proofs.

Let $X_0 \subseteq \mathbf{R}^n$ be a variety (hence Zariski-closed), and let

$$I_0 = \{p \in \mathbf{R}[X_1, \ldots, X_n] : p|X_0 \equiv 0\}.$$

Then X, the Zariski-closure of X_0 in $\mathbf{C}^n$, is the set of common zeros of $I_0 \subset \mathbf{C}[X_1, \ldots, X_n]$; indeed, the polynomials vanishing on X are just the elements of $I = I_0 \otimes_{\mathbf{R}} \mathbf{C}$. (Simply notice that any $p \in \mathbf{C}[X_1, \ldots, X_n]$ can be written uniquely as $p_1 + ip_2$, with $p_j \in \mathbf{R}[X_1, \ldots, X_n]$, and that $p|X_0 \equiv 0 \Leftrightarrow p_1|X_0, p_2|X_0$ are both 0.)

Let $X \subseteq \mathbf{C}^n$ be an irreducible algebraic variety. Then X is a finite disjoint union of complex manifolds, $X = S_1 \cup \cdots \cup S_k$, where each S_j is Zariski-open in the complement of $X \backslash (S_1 \cup \cdots \cup S_{j-1})$ for all j. We define $\dim_{\mathbf{C}} X$ to be the maximum of the $\dim_{\mathbf{C}} S_i$. This coincides with the usual algebraic definition of the dimension of a variety, and is independent of the decomposition of X given above. One useful fact is

(3.2) Any subvariety of X has lower dimension than X.
(See, e.g., [**4**, p. 6]). Thus, in particular, we have

$$\dim X = \dim S_1 > \dim(S_2 \cup \cdots \cup S_k).$$

For algebraic varieties in $\mathbf{R}^n$, there are similar results. We shall need a more general fact. A *semialgebraic set* in $\mathbf{R}^n$ is a set S in the Boolean algebra generated by the sets $\{x : p(x) \geq 0\}$, $\{x : p(x) = 0\}$, where p is a polynomial. (Observe that for any given semialgebraic set, only finitely many polynomials play a role.) Given semialgebraic sets $T_1, \ldots, T_r$, a *stratification* of $(T_1, \ldots, T_r)$ is a partition $\mathcal{P} = \{S_1, \ldots, S_m\}$ of $\mathbf{R}^n$ such that
(i) each S_i is a connected embedded manifold in $\mathbf{R}^n$ (the manifold topology = the relative topology);
(ii) for every $x \in \mathbf{R}^n$, $\exists$ a neighborhood N_x of $\mathbf{R}^n$ such that $N_x \cap S_i$ is connected (or empty), $1 \leq i \leq m$;
(iii) $S_i \cap S_j^- \neq \varnothing \Rightarrow S_i \subseteq S_j^-$ (the closure is in the Euclidean topology);
(iv) each S_j is semialgebraic;
(v) each T_i is the union of the S_j meeting it.
Such stratifications always exist; see [**8**]. It is not hard to see that if $\mathcal{P}$ is a stratification of the semialgebraic set T, then $\max(\dim S_j : S_j \in \mathcal{P})$ is independent of $\mathcal{P}$; we define $\dim_{\mathbf{R}} T$ to be this number. Furthermore, there is a unique natural measure class $[\nu]$ on T. This is obtained as follows: let $\dim T = k$. On each S_j with $\dim S_j = k$, use a nonvanishing k-form to get a measure ν_j; if $\dim S_j < k$, define

ν_j on S_j to be 0. Now let $\nu = \sum \nu_j$. It is easy to see that $[\nu]$ is independent of the choice of k-forms and of the stratification.

The following proposition lets us compare dimensions; it is close to one in Appendix IV of [10].

(3.3) PROPOSITION. *Let X_0 be an irreducible algebraic variety in $\mathbf{R}^n$; let X be its Zariski-closure in $\mathbf{C}^n$. Then*
(i) *X is irreducible;*
(ii) *any subvariety of X or X_0 has lower dimension;*
(iii) *$\dim_{\mathbf{R}} X_0 = \dim_{\mathbf{C}} X$.*

PROOF. We noted (i) in Lemma 3.1. Statement (ii) for X is just (3.2); by (3.1)(ii) it follows for varieties over $\mathbf{R}$ once we prove (iii). Thus we need only prove (iii).

In view of (ii), the decomposition of X into complex manifolds has a single piece U of largest dimension; set $m = \dim_{\mathbf{C}} U$. If $U \cap X_0 = \varnothing$, then $X \backslash U$ is Zariski-closed and contains X_0; this contradicts the definition of X as the Zariski-closure of X_0. Let $x \in U \cap X_0$, and let I be the prime ideal of polynomials vanishing on X. From Corollary 1.20 of [4], there are polynomials $f_1, \ldots, f_k \in I$ $(k = n - m)$ such that the f_j have linearly independent linear terms and

$$I = \left\{ f \in \mathbf{C}[X] : f = \sum_{j=1}^{k} \frac{h_j}{g} f_j, h_j, g \in \mathbf{C}[X_1, \ldots, X_n], g(x) \neq 0 \right\}.$$

We may assume that the linear terms of the f_j are real; from the proof in [4] we may further assume (taking real parts) that the f_j are in $\mathbf{R}[X_1, \ldots, X_n]$. The Implicit Function Theorem now gives a real manifold of dimension m in $X \cap \mathbf{R}^n = X_0$ and containing x; hence $\dim_{\mathbf{R}} X_0 \geq \dim_{\mathbf{C}} X$. The other inequality is also an easy consequence of the Implicit Function Theorem. $\square$

Let $S \subseteq \mathbf{F}^m$; let $\varphi: S \to \mathbf{F}^n$ be a map. We say that φ is *rational nonsingular* if there are Zariski-open sets $U_\theta \subseteq \mathbf{F}^m$ and explicit rational maps φ_θ on U_θ,

$$\varphi_\theta(z_1, \ldots, z_m) = \varphi_\theta(z) = \left(\frac{r_1(z)}{s_1(z)}, \ldots, \frac{r_n(z)}{s_n(z)} \right),$$
$$r_i, s_i \in \mathbf{F}[z_1, \ldots, z_m], \ s_i \neq 0 \text{ on } U_\theta,$$

such that (i) $\varphi_\theta = \varphi$ on $S \cap U_\theta$ and (ii) the U_θ cover S.

(3.4) LEMMA. *Let $S \subseteq \mathbf{F}^m$; let $\varphi: S \to \mathbf{F}^n$ be rational nonsingular on S, and let $T = \varphi(S)$. Then $\varphi: S \to T$ is continuous when both sets have the relative Zariski topologies.*

PROOF. We begin with two reductions.
(i) It suffices to show that for any rational nonsingular map $\varphi: S \to \mathbf{F}^n$, $\varphi^{-1}(0)$ is Zariski-closed in S. For if A is any Zariski-closed set in T, then there are finitely many polynomials P_j, $1 \leq j \leq d$, such that A is the set of common zeros of the p_j. Let $p = (p_1, \ldots, p_d): \mathbf{F}^n \to \mathbf{F}^d$; then $p \circ \varphi$ is rational nonsingular on S, and $\varphi^{-1}(A) = (p \circ \varphi)^{-1}(0)$.

Thus we let $X = \varphi^{-1}(0) \subseteq S$; let $X^{\#}$ be the Zariski closure of X in $\mathbf{F}^m$. It suffices to prove

(ii) If $X^{\#}$ is irreducible, then X is Zariski-closed in S.

For we can write $X^{\#} = A_1 \cup \cdots \cup A_r$ with the A_j Zariski-closed and irreducible, and $A_i \not\subset A_j$ if $i \neq j$. Set $X_j = X \cap A_j$; we may assume that each X_j is nonempty. Then $\bigcup_j X_j^{\#} = X^{\#}$, since it is closed, covers X, and is contained in $X^{\#}$. For fixed i, we thus have $\bigcup_j X_j^{\#} \cap A_i = A_i$. As A_i is irreducible, we must have $X_j^{\#} \cap A_i = A_i$ for some j; $j = i$, since $X_j^{\#} \cap A_i \subseteq A_j \cap A_i \neq A_i$ if $j \neq i$. So $X = \bigcup_j X_j$, where each $X_j^{\#}$ is irreducible. Let $g_1, \ldots, g_s$ be polynomials on $\mathbf{F}^m$ such that $X_1^{\#}$ is the set of common zeros of the g_j; let $\tilde{\varphi} = (\varphi; g_1, \ldots, g_s): \mathbf{F}^m \to \mathbf{F}^{n+s}$. Now $S \cap \tilde{\varphi}^{-1}(0) = X_1$; $X_1^{\#}$ is irreducible, and (ii) implies that X_1 is Zariski-closed in S. Similarly, the other X_j are Zariski-closed in S.

We now prove (ii); thus we assume that $X^{\#}$ is irreducible. Let φ_θ, U_θ cover S as above, and let $I = \{\theta : U_\theta \cap X \neq \varnothing\}$. Fix $\theta \in I$ and write $\varphi_\theta = (r_1/s_1, \ldots, r_n/s_n)$ with $s_i \neq 0$ on U_θ and $\varphi_\theta = \varphi$ on $S \cap U_\theta$. Let A_θ be the set of common zeros of the r_i. Then $X \cap U_\theta = S \cap U_\theta \cap A_\theta$. Clearly $X \subseteq \bigcup_{\theta \in I} A_\theta$ and the A_θ are Zariski-closed in $\mathbf{F}^m$. By the irreducibility of $X^{\#}$, there is some θ_0 with $X \subseteq A_{\theta_0}$. Suppose that $\theta \in I$; then $X \cap U_\theta \cap U_{\theta_0}$ is nonempty Zariski-dense in $X^{\#}$. Let

$$\varphi_{\theta_0} = (r_1/s_1, \ldots, r_n/s_n), \qquad \varphi_\theta = (r_1'/s_1', \ldots, r_n'/s_n').$$

Then on $X \cap U_\theta \cap U_{\theta_0}$, we have

(3.5)   (a) $r_i s_i' = r_i' s_i = 0$,   all i;
                           (b) $s_i \neq 0$, $s_i' \neq 0$,   all i.

Then (3.5a) holds in $X^{\#}$, by density. Because $X \subseteq A_{\theta_0}$, the r_i are identically 0 on X; hence the r_i' are 0 on $X \cap U_\theta \cap U_{\theta_0}$. Therefore

$$\varnothing \neq (X \cap U_\theta \cap U_{\theta_0}) \subseteq A_\theta, \qquad \forall \theta \in I;$$

the irreducibility of $X^{\#}$ gives

(3.6) $(X \cap U_\theta \cap U_{\theta_0})^{\#} = X^{\#} \subseteq A_\theta, \qquad \forall \theta \in I.$

We also have

$$S \cap A_\theta \cap U_\theta \subseteq X, \qquad \forall \theta \in I,$$

as noted above.

Now let $x \in X^{\#} \cap S$. Since $X \cap U_\theta \neq \varnothing \Leftrightarrow X^{\#} \cap U_\theta \neq \varnothing$, we have $X^{\#} \subseteq \bigcup_{\theta \in I} U_\theta$. Thus there exists $\theta \in I$ with $x \in U_\theta$. From (3.6), $x \in A_\theta$. Thus $x \in A_\theta \cap U_\theta \cap S \subseteq X$, so that $X^{\#} \cap S \subseteq X$. This proves that X is Zariski-closed in S. $\square$

We note two corollaries.

(3.7) COROLLARY. *If $\varphi: S \to T$ is birational nonsingular, then it is a Zariski homeomorphism.*

(3.8) COROLLARY. *If $\varphi: S \to \mathbf{F}^n$ is rational nonsingular, and if $S^{\#} \subseteq \mathbf{F}^m$ is irreducible, then $\varphi(S)^{\#}$ is irreducible.*

PROOF. Let $\varphi(s) = T$. If $T^{\#}$ is not irreducible, then $T = X_1 \cup X_2$ with X_1, X_2 Zariski-closed in T. In that case, $\varphi^{-1}(X_1)$, $\varphi^{-1}(X_2)$ show that S is not irreducible. $\square$

We shall also find the following lemmas useful.

(3.9) LEMMA. *Suppose that* $\varphi: S \to T$ *is a birational nonsingular map* ($S \subseteq \mathbf{F}^m$, $T \subseteq \mathbf{F}^n$). *Assume that*

(a) T *is Zariski-open in its closure:* $T = B^c \cap T^\#$ *for some Zariski-closed* $B \subseteq \mathbf{F}^n$ (B^c = *complement of* B);

(b) *there is an irreducible closed variety* $E \subseteq \mathbf{F}^m$ *such that* $S \cap E$ *contains a nonempty Zariski-open subset of* E.

Let $W = \varphi(S \cap E)$. *Then* $W^\#$ *is irreducible in* $\mathbf{F}^n$ *and* W *contains a subset* A *that is Zariski-open and dense in* $W^\#$.

PROOF. The restriction of φ to $S \cap E$ is rational nonsingular; $(S \cap E)^\# = E$ is irreducible by assumption. Corollary 3.8 now says that $W^\#$ is irreducible. Let $\psi = \varphi^{-1}: T \to S$; let $p = (p_1, \ldots, p_d): \mathbf{F}^m \to \mathbf{F}^d$ be polynomial, with $p^{-1}(0) = E$. Because ψ is rational nonsingular, there are Zariski-open subsets $U_\theta \subseteq \mathbf{F}^n$ covering T, together with rational maps ψ_θ, nonsingular on U_θ, such that $\psi_\theta = \psi$ on $T \cap U_\theta$. Then $p \circ \psi: T \to \mathbf{F}^d$ is rational nonsingular and is given on $T \cap U_\theta$ by $g_\theta = p \circ \psi_\theta$:

$$g_\theta = \left(\frac{r_1}{s_1}, \ldots, \frac{r_d}{s_d} \right) \quad \text{on } U_\theta \text{ with } s_i \neq 0 \text{ on } U_\theta; r_i, s_i \in \mathbf{F}[t_1, \ldots, t_n].$$

Let $A_\theta = \{ x \in \mathbf{F}^n : r_i(x) = 0, 1 \leq i \leq d \}$ and $G_\theta = \{ x \in U_\theta : g_\theta(x) = 0 \}$. Then A_θ is Zariski-closed in $\mathbf{F}^n$ and $G_\theta = A_\theta \cap U_\theta$. Now

$$W = (p \circ \psi)^{-1}(0) = T \cap (p \circ \psi)^{-1}(0) \qquad (T = \mathrm{dom}\,\psi);$$
$$W \cap U_\theta = U_\theta \cap (p \circ \psi)^{-1}(0) = G_\theta \cap U_\theta \cap T = A_\theta \cap U_\theta \cap T.$$

We know from (i) that $T = B^c \cap T^\#$ for some Zariski-closed $B \subseteq \mathbf{F}^n$. Hence

(3.10) $\qquad W \cap U_\theta = B^c \cap T^\# \cap G_\theta \cap U_\theta = B^c \cap T^\# \cap A_\theta \cap U_\theta.$

Let $I = \{ \theta : U_\theta \text{ meets } W \} = \{ \theta : U_\theta \text{ meets } W^\# \}$. Then for $\theta \in I$,

$$W \cap U_\theta = T \cap (p \circ \psi)^{-1}(0) \cap U_\theta;$$
$$W^\# = (W \cap U_\theta)^\# \subseteq T^\# \cap ((p \circ \varphi)^{-1}(0) \cap U_\theta)^\# = T^\# \cap G_\theta^\# \subseteq T^\# \cap A_\theta.$$

Hence

(3.11) $\qquad W^\# \cap U_\theta \subseteq T^\# \cap A_\theta \cap U_\theta, \qquad \forall \theta \in I.$

Comparing (3.10) and (3.11) gives

$$W^\# \cap U_\theta \cap B^c \subseteq B^c \cap T^\# \cap A_\theta \cap U_\theta = W \cap U_\theta \subseteq W^\# \cap U_\theta, \qquad \forall \theta \in I.$$

Since the U_θ cover $W^\#$, we get (summing over $\theta \in I$) $W^\# \cap B^c = W \subseteq W^\#$. So we can take $A = W^\# \cap B^c = W$. □

We conclude this section by recalling an alternative version of the Multiplicity Theorem 2.2, valid in the case where $\dim_\chi = 1$. It is simpler to prove than Theorem 2.2 because it stops short of describing multiplicities canonically. It is essentially established as Theorem 4 of [2], well before one meets the most difficult technical details involved in Theorem 2.2.

(3.12) THEOREM (NONCANONICAL MULTIPLICITY FORMULA). *Let* $\mathfrak{k}_0$ *be a Lie subalgebra of a real Lie algebra* $\mathfrak{g}_0$, *and let* $K_0 \subseteq G_0$ *be the corresponding Lie group. Suppose that* $\chi = e^{2\pi i f} \in \widehat{K_0}$ *is a 1-dimensional unitary representation;*

let $\rho = \mathrm{Ind}(K \uparrow G, \chi)$. *Fix a Malcev basis in* $\mathfrak{g}_0$ *and form the corresponding $\mathcal{E}$-layering of* $\mathfrak{g}_0^*$. *Let* $e \in \mathcal{E}$ *be the first index such that* U_e *meets* $f' + \mathfrak{k}_0^\perp$; *and let* $P_e: U_e \to \Sigma_e \times V_{S(e)}$ *be the (rational) inverse of the parametrizing map. Then*

(i) e *is also the first index such that* U_e *meets* $f' + E_1^0$, *and their intersection is Zariski-open in* $f' + E_1^0$. *(The subspace E_1^0 is the one of Proposition 2.5.)*

Let $S_1^0 = U_e \cap (f' + E_1^0)$; *define* $\varphi: S_1^0 \to \Sigma_e \subseteq V_{T(e)}$ *by* $\varphi = \pi_T \circ P_e|_{S_1^0}$, *where* π_T *is the projection onto* $V_{T(e)}$ *killing* $V_{S(e)}$. *Then*

(ii) $\Sigma_1^0 = \varphi(S_1^0)$ *is semialgebraic. If* $[\nu_1]$ *is its canonical measure class, then*

$$(3.13) \qquad \rho \cong \int_{\Sigma_1^0}^{\oplus} n(l)\pi_l \, d\nu_1(l),$$

where either $n(l) = +\infty$ *(when* $\tau_0 > 0$*) or*

$$(3.14) \qquad n(l) = \mathrm{Card}(G_0 \cdot l \cap (f' + E_1^0)), \quad all \ l \in \Sigma_1^0 \ (when \ \tau_0 = 0).$$

In the latter case, $n(l)$ *is uniformly bounded,* ν_1*-almost everywhere on* Σ_1^0.

REMARKS. 1. The result is a geometric multiplicity formula, but the variety $f' + E_1^0$ is not canonically defined.

2. If $\tau_0 > 0$, then $\mathrm{Card}(G_0 \cdot l \cap (f' + E_1^0)) = \infty$; thus (3.14) holds regardless of the size of τ_0. (See Theorem 2.2 for the definition of τ_0.)

4. This section is devoted to various results on coadjoint orbits for complex nilpotent Lie groups and algebras. Many of the proofs are essentially the same as for real groups (for these we refer to [2]), and many others are elementary consequences of the definitions. We often use a subscript "zero" to indicate a real Lie algebra.

Let $\mathfrak{g}$ be a complex nilpotent Lie algebra. There is a corresponding simply connected complex Lie group G, which acts on $\mathfrak{g}$ via the adjoint action. If $\mathfrak{g}_{\mathbf{R}} = \mathfrak{g}$ with the field reduced to $\mathbf{R}$, $\mathrm{Ad}\, G$ acts on $\mathfrak{g}_{\mathbf{R}}$ as well. There are two corresponding contragredient actions $\mathrm{Ad}^*(G)$, one on $\mathfrak{g}^* = \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathbf{C})$ via $\mathbf{C}$-linear maps, and one on $\mathfrak{g}_{\mathbf{R}}^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{g}_{\mathbf{R}}, \mathbf{R})$ via $\mathbf{R}$-linear maps.

(4.1) LEMMA. *Define* $J: \mathfrak{g}_{\mathbf{R}}^* \to \mathfrak{g}^*$ *by*

$$(4.2) \qquad\qquad (Jf)(Z) = f(Z) - if(iZ), \qquad Z \in \mathfrak{g}.$$

Then J *is an $\mathbf{R}$-linear isomorphism that intertwines the actions of* $\mathrm{Ad}^*(G)$. *Its inverse is given by*

$$(J^{-1}\varphi)(Z) = \mathrm{Re}(\varphi(Z)), \qquad \forall \varphi \in \mathfrak{g}^*, \ \forall Z \in \mathfrak{g}_{\mathbf{R}}.$$

PROOF. If $\varphi \in \mathfrak{g}^*$, write $\varphi = \varphi_1 + i\varphi_2$, where $\varphi_1(Z) = \mathrm{Re}\,\varphi(Z)$ and $\varphi_2(Z) = \mathrm{Im}\,\varphi(Z)$. Then $\varphi_1, \varphi_2 \in \mathfrak{g}_{\mathbf{R}}^*$. Moreover,

$$\varphi_1(iZ) + i\varphi_2(iZ) = \varphi(iZ) = i\varphi(Z) = -\varphi_2(Z) + i\varphi_1(Z),$$

so that

$$\varphi_2(Z) = -\varphi_1(iZ), \qquad \forall Z \in \mathfrak{g}.$$

This shows that $\varphi = J\varphi_1$, which proves the second claim. If $x \in G$ and $z \in \mathfrak{g}$, then

$$((\mathrm{Ad}^*x)^{-1}\varphi)(Z) = \varphi((\mathrm{Ad}\,x)Z) = \varphi_1((\mathrm{Ad}\,x)Z) - i\varphi_1(i(\mathrm{Ad}\,x)Z)$$

$$= \varphi_1((\mathrm{Ad}\,x)Z) - i\varphi_1((\mathrm{Ad}\,x)(iZ))$$

$$= (\mathrm{Ad}^*x^{-1}(\varphi_1))(Z) - i(\mathrm{Ad}^*x^{-1}(\varphi_1))(iZ)$$

$$= (J(\mathrm{Ad}^*x^{-1})\varphi_1)(Z).$$

As $\varphi = J\varphi_1$, the intertwining property is proved. $\square$

The standard proof (in, e.g., [5]) shows that for $\varphi \in \mathfrak{g}^*, \mathrm{Stab}_G(\varphi) = \{x \in G: \mathrm{Ad}^*(x)\varphi = \varphi\}$ is a connected Lie subgroup of G whose Lie algebra is $\mathfrak{p}(\varphi) = \{Z \in \mathfrak{g}: \varphi([Z, \mathfrak{g}]) = (0)\}$. If $f \in \mathfrak{g}_{\mathbf{R}}^*$, then its stabilizer has Lie algebra $\mathfrak{p}(f) = \{Z \in \mathfrak{g}_{\mathbf{R}}: f([Z, \mathfrak{g}_{\mathbf{R}}]) = (0)\}$. A simple calculation gives

$$(4.3) \qquad \mathfrak{p}(Jf) = \mathfrak{p}(f), \qquad \forall f \in \mathfrak{g}_{\mathbf{R}}.$$

Hence $\mathfrak{p}(f)$ is always a complex subalgebra of $\mathfrak{g}$ ($= \mathfrak{g}_{\mathbf{R}}$). Construct a maximal isotropic complex subalgebra $\mathfrak{m}$ for Jf. Simple calculations with dimensions prove $\mathfrak{m}_{\mathbf{R}}$ maximal isotropic for f; that is,

(4.4) For all $f \in \mathfrak{g}_{\mathbf{R}}^*$, f has a maximal isotropic subalgebra that is complex.

This means that every element of $G^\smallfrown$ is induced from a 1-dimensional representation of a complex subgroup. Note also that $\mathrm{codim}_{\mathbf{C}}\mathfrak{p}(f)$ is even, so that $\dim_{\mathbf{R}}(\mathfrak{g}_{\mathbf{R}}/\mathfrak{p}(f))$ is divisible by 4.

We next consider a different situation: let $\mathfrak{g}_0$ be a real nilpotent Lie algebra, with corresponding Lie group G_0, and let $\mathfrak{g}$ be its complexification. Imbed $\mathfrak{g}_0^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{g}_0, \mathbf{R})$ into $\mathfrak{g}^*$ in the natural way via $I: \mathfrak{g}_0^* \to \mathfrak{g}^*$:

$$I(l)(Z) = I(l)(X + iY) = l(X) + il(Y), \qquad l \in \mathfrak{g}_0^*, \ Z = X + iY \in \mathfrak{g}.$$

Then $\mathrm{Range}\, I = \{\varphi \in \mathfrak{g}^*: \varphi(\mathfrak{g}_0) \subseteq \mathbf{R}\}$. If we imbed $\mathfrak{g}_0^*$ into $\mathfrak{g}_{\mathbf{R}}^*$ via

$$(4.5) \qquad (I_{\mathbf{R}}l)(Z) = (I_{\mathbf{R}}l)(X + iY) = l(X),$$

then $I = J \circ I_{\mathbf{R}}$ and $\mathrm{Range}\, I_{\mathbf{R}} = (i\mathfrak{g}_0)^\perp \subseteq \mathfrak{g}_{\mathbf{R}}^*$. Regard G_0 as a subgroup of $G = \exp(\mathfrak{g})$; then $I_{\mathbf{R}}, J$, and I all commute with the coadjoint actions of G_0. Furthermore, if $l \in \mathfrak{g}_0^*$, then $\mathfrak{p}_l = \{X \in \mathfrak{g}_0: l([X, \mathfrak{g}_0]) = (0)\}$ satisfies

$$\mathfrak{p}(I\, l) = \mathfrak{p}_l \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{p}_l + i\mathfrak{p}_l = \mathfrak{p}(I_{\mathbf{R}}l).$$

If $\mathfrak{m}_0$ is a polarizing subalgebra for l, then $\mathfrak{m} = \mathfrak{m}_0 + i\mathfrak{m}_0$ is polarizing for both Il and $I_{\mathbf{R}}l$.

We now compare the $\mathcal{E}$-layerings for $\mathfrak{g}^*$ and $\mathfrak{g}_{\mathbf{R}}^*$ under the action of G. Let $\{X_1, \ldots, X_n\}$ be a (complex) Malcev basis for $\mathfrak{g}$, with dual basis $X_1^*, \ldots, X_n^*$ in $\mathfrak{g}^*$. Then $\{Y_1, \ldots, Y_{2n}\}$, where $Y_{2j-1} = X_j$ and $Y_{2j} = iX_j$, is a (real) Malcev basis for $\mathfrak{g}_{\mathbf{R}}$, and the dual basis $Y_1^*, \ldots, Y_{2n}^*$ is a Jordan-Hölder basis for $\mathfrak{g}_{\mathbf{R}}^*$. Moreover, calculation gives

$$(4.6) \qquad J(Y_{2j-1}^*) = X_j^*, \quad J(Y_{2j}^*) = -iX_j^*, \qquad 1 \le j \le n.$$

Let

$$\mathfrak{g}_j^* = \mathbf{C}\text{-span}\{X_{j+1}^*, \ldots, X_n^*\}, \quad p_j: \mathfrak{g}^* \to \mathfrak{g}^*/\mathfrak{g}_j^*, \qquad 1 \le j \le n;$$

$$(\mathfrak{g}_{\mathbf{R}}^*)_j = \mathbf{R}\text{-span}\{Y_{j+1}^*, \ldots, Y_n^*\}, \quad p_j^{\mathbf{R}}: \mathfrak{g}_{\mathbf{R}}^* \to \mathfrak{g}_{\mathbf{R}}^*/(\mathfrak{g}_{\mathbf{R}}^*)_j, \qquad 1 \le j \le 2n.$$

Then (4.6) says that $J((\mathfrak{g}_{\mathbf{R}}^*)_{2j}) = \mathfrak{g}_j^*$. We thus get induced J-maps $J_{2j}: \mathfrak{g}_{\mathbf{R}}^*/(\mathfrak{g}_{\mathbf{R}}^*)_{2j} \to \mathfrak{g}^*/\mathfrak{g}_j^*$; these maps are equivariant for the induced action of $\mathrm{Ad}^*(G)$, and we have $p_j \circ J = J_{2j} \circ p_{2j}^{\mathbf{R}}$.

For $l \in \mathfrak{g}^*$, define $e_j(l) = \dim_{\mathbf{C}} G \cdot p_j(l)$ for $1 \le j \le n$, and let $e(l) = (e_1(l), \ldots, e_n(l))$; let $\mathcal{E} = \{e: \exists l \in \mathfrak{g}^* \text{ with } e(l) = e\} \subset \mathbf{Z}^n$. Similarly, for $f \in \mathfrak{g}_{\mathbf{R}}^*$, define $e_j(f) = \dim_{\mathbf{R}} G \cdot p_j^{\mathbf{R}}(l)$ for $1 \le j \le 2n$; set $e(f) = (e_1(f), \ldots, e_{2n}(f))$, and let $\mathcal{E}_{\mathbf{R}} = \{e: \exists f \in \mathfrak{g}_{\mathbf{R}}^* \text{ with } e(f) = e\} \subset \mathbf{Z}^{2n}$.

(4.7) LEMMA. *Let notation be as above. For $e \in \mathcal{E}_{\mathbf{R}}$, define $\alpha(e) \in \mathbf{R}^n$ by $\alpha(e)_j = \frac{1}{2} e_{2j}$. Then $\alpha(e) \in \mathcal{E}$, and the map $e \mapsto \alpha(e)$ is a bijection of $\mathcal{E}_{\mathbf{R}}$ onto $\mathcal{E}$. If $e' = (e_1, \ldots, e_n) \in \mathcal{E}$, then $\alpha^{-1}(e') = (e_1, \ldots, e_{2n})$, with $e_{2j} = 2e_j'$, $e_{2j-1} = e_j' + e_{j-1}'$ (here $e_0' = 0$). Moreover,*

$$(4.8) \qquad\qquad e(Jf) = \alpha(e(f)), \qquad \forall f \in \mathfrak{g}_{\mathbf{R}}^*.$$

PROOF. We have

$$G \cdot p_j(Jf) = J(G \cdot p_{2j}^{\mathbf{R}}(f)), \qquad \forall f \in \mathfrak{g}_{\mathbf{R}}^* \text{ and } \forall j,\ 1 \le j \le n,$$

as noted earlier. Hence

$$e_j(Jf) = \dim_{\mathbf{C}}(G \cdot p_j(Jf)) = \tfrac{1}{2}\dim_{\mathbf{R}}(G \cdot p_j(Jf)) = \tfrac{1}{2}\dim_{\mathbf{R}}(G \cdot p_{2j}^{\mathbf{R}}(f)) = \tfrac{1}{2}e_{2j}(f),$$

from which (4.8) follows. This implies that α is a bijection. To verify the formula $\alpha^{-1}(e')$, note that $e_{2j} = 2e_j'$ by the formula for α; as for the odd terms, their value is determined by the fact that $e_{i+1} - e_i = 0$ or 1 for all i if $e \in \mathcal{E}_{\mathbf{R}}$ (see Theorem 2.1).   □

It follows from (4.8) that if we define layers $U_e^{\mathbf{R}}, U_e$ in the usual way for e in $\mathcal{E}^{\mathbf{R}}, \mathcal{E}$, then

$$(4.9) \qquad\qquad J(U_e^{\mathbf{R}}) = U_{\alpha(e)}, \qquad \forall e \in \mathcal{E}^{\mathbf{R}}.$$

Furthermore, Theorem 2.1(ii) refers to an ordering of indices in $\mathcal{E}^{\mathbf{R}}$. This ordering is explicitly described in [2], or in §III.1.1 of [1]. There is a similar ordering for $\mathcal{E}$, and it is not hard to see that one can choose the orderings so that α is order-preserving.

Now fix $e \in \mathcal{E}^{\mathbf{R}}$ and let $\varepsilon = \alpha(e)$. Consider the rational nonsingular parametrizing maps of Theorem 2.1:

$$\Sigma_\varepsilon \times V_{S(\varepsilon)} \underset{Q_e}{\overset{P_e}{\leftrightarrows}} U_\varepsilon, \qquad \Sigma_e \times V_{S(e)} \underset{Q_{\mathbf{R},e}}{\overset{P_{\mathbf{R},e}}{\leftrightarrows}} U_e^{\mathbf{R}}.$$

From the formula for α^{-1}, we see that

$$S(\varepsilon) = \{j_1, \ldots, j_k\} \Leftrightarrow S(e) = \{2j_1 - 1, 2j_1, \ldots, 2j_k - 1, 2j_k\},$$

and similarly for $T(e), T(\varepsilon)$. By (4.6),

$$V_{S(\varepsilon)} = J(V_{S(e)}), \qquad V_{T(\varepsilon)} = J(V_{T(e)}).$$

Since $J(U_e^{\mathbf{R}}) = U_\varepsilon$, we get

$$\Sigma_\varepsilon = U_\varepsilon \cap V_{T(\varepsilon)} = J(U_e \cap V_{T(e)}) = J(\Sigma_e).$$

In fact, we have $P_e \circ J = J \circ P_{\mathbf{R},e}$; this follows from the geometric description of P_e and $P_{\mathbf{R},e}$,

$$P_e(l) = (G \cdot l \cap V_{T(e)},\ \pi_S(l))$$

(and similarly for $P_{\mathbf{R},e}$), given in Theorem 2.1.

5. In this section, we prove Theorem 1.1. We use the notation of §§3 and 4.

From (4.4), we may assume that χ is 1-dimensional, $\chi = e^{2\pi i f_0}$ on K, where $f_0 \in \mathfrak{k}_{\mathbf{R}}^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{k}_{\mathbf{R}}, \mathbf{R})$ is a Lie algebra homomorphism. It is easy to verify that $\varphi_0 = Jf_0 \in \mathrm{Hom}_{\mathbf{C}}(\mathfrak{k}, \mathbf{C})$ is a Lie algebra homomorphism. It is also easy to verify that the J-maps have the following properties:

(i) If $P, P_{\mathbf{R}}$ are the natural projections, then

(5.1) $$\begin{array}{ccc} \mathfrak{g}_{\mathbf{R}}^* & \xrightarrow{\ J\ } & \mathfrak{g}^* \\ P_{\mathbf{R}}\downarrow & & \downarrow P \\ \mathfrak{k}_{\mathbf{R}}^* & \xrightarrow{\ J_K\ } & \mathfrak{k}^* \end{array} \qquad \text{is commutative.}$$

(ii) If $\mathfrak{k}_{\mathbf{R}}^{\perp} = \{f \in \mathfrak{g}_{\mathbf{R}}^*: f|_{\mathfrak{k}_{\mathbf{R}}} \equiv 0\}$, $\mathfrak{k}^{\perp} = \{\varphi \in \mathfrak{g}^*: \varphi|_{\mathfrak{k}} \equiv 0\}$, then $J(\mathfrak{k}_{\mathbf{R}}^{\perp}) = \mathfrak{k}^{\perp}$.

(iii) If S is any subset of $\mathfrak{k}_{\mathbf{R}}^*$, then $JP_{\mathbf{R}}^{-1}(S) = P^{-1}J_K(S)$. In particular, if $f' \in \mathfrak{g}_{\mathbf{R}}^*$ satisfies $f'|_{\mathfrak{k}} = f_0$, and if similarly $\varphi'|_{\mathfrak{k}} = \varphi_0$, then $\mathcal{O}_{\chi} = \{f_0\}$ and J maps $P_{\mathbf{R}}^{-1}(\mathcal{O}_{\chi}) = f' + \mathfrak{k}_{\mathbf{R}}^{\perp}$ to $P^{-1}(\varphi_0) = \varphi' + \mathfrak{k}^{\perp}$. These sets are K-invariant, and J maps K-orbits to K-orbits.

We need to modify the Multiplicity Theorem 3.12 so that it refers to elements of $\mathfrak{g}^*$ rather than elements of $\mathfrak{g}_{\mathbf{R}}^*$. Let U_e be the first layer of $\mathfrak{g}_{\mathbf{R}}^*$ that meets $f' + \mathfrak{k}_{\mathbf{R}}^{\perp}$, and let $\alpha(e) = \varepsilon$. Then (4.9) and (iii) above imply that U_ε is the first layer of $\mathfrak{g}^*$ meeting $\varphi' + \mathfrak{k}^{\perp}$. To use Theorem 3.12, however, we also need to describe $J(f' + E_1^{\mathbf{R}})$, where $E_1^{\mathbf{R}}$ is the subspace of $\mathfrak{k}_{\mathbf{R}}^*$ appearing in the multiplicity formula. This subspace was produced by examining a different layering, adapted to the pair $\mathfrak{k}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}$. Here are the details we need.

Comparison of $\mathcal{D}$-layerings. Starting with $\mathfrak{k} \subseteq \mathfrak{g}$, we can find complex subalgebras $\mathfrak{k} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_r = \mathfrak{g}$ with $\dim_{\mathbf{C}}(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1$. Choose $X_j \in \mathfrak{g}_j \backslash \mathfrak{g}_{j-1}$ and define dual vectors $X_j^* \in \mathfrak{g}^*$ such that $X_j^*(X_k) = \delta_{jk}$, $X_j^*(\mathfrak{k}) = (0)$. Thus the X_j^* form a (complex) basis for $\mathfrak{k}^{\perp}$. Let $G_j = \exp(\mathfrak{g}_j)$ be the corresponding Lie subgroups to the $\mathfrak{g}_j$ in G. Then define vectors $Y_1, \ldots, Y_{2r}$ in $\mathfrak{g}_{\mathbf{R}}$ by $Y_{2j-1} = X_j$, $Y_{2j} = iX_j$. These form an $\mathbf{R}$-basis for $\mathfrak{g}_{\mathbf{R}}$ above $\mathfrak{k}_{\mathbf{R}}$, and, for $1 \le j \le 2r$,

$$\mathfrak{h}_j = \mathfrak{k}_{\mathbf{R}} + \mathbf{R}\text{-span}\{Y_1, \ldots, Y_j\}$$

is a subalgebra of $\mathfrak{g}_{\mathbf{R}}$, with $\dim_{\mathbf{R}}(\mathfrak{h}_j/\mathfrak{h}_{j-1}) = 1$. Let $H_j = \exp \mathfrak{h}_j$. Then $(\mathfrak{g}_j)_{\mathbf{R}} = \mathfrak{h}_{2j}$ as real Lie algebras, and $\mathfrak{h}_{2j}^* = (\mathfrak{g}_j)_{\mathbf{R}}^*$. If $p_j: \mathfrak{g}^* \to \mathfrak{h}_j^*$ and $p_j^{\mathbf{R}}: \mathfrak{g}_{\mathbf{R}}^* \to \mathfrak{h}_j^*$ are the natural projections, and if $J_j: (\mathfrak{g}_j)_{\mathbf{R}}^* \to \mathfrak{g}_j^*$ is defined similarly to J, then we have $p_j \circ J = J_j \circ p_{2j}^{\mathbf{R}}$. Moreover, the map J_j intertwines the actions of $G_j = H_{2j}$ on $\mathfrak{g}_j^*$, $\mathfrak{h}_{2j}^*$, and we have

(5.2) $\dim_{\mathbf{C}} G_j \cdot p_j(Jf) = \frac{1}{2}\dim_{\mathbf{R}} G_j \cdot p_{2j}^{\mathbf{R}}(f) = \frac{1}{2}\dim_{\mathbf{R}} H_{2j} \cdot p_{2j}^{\mathbf{R}}(f)$, $\qquad \forall f \in \mathfrak{g}_{\mathbf{R}}^*.$

Let $d_j(l) = \dim_{\mathbf{C}} G_j \cdot p_j(l)$ for $l \in \mathfrak{g}^*$ and $1 \le j \le r$; set $d(l) = (d_1(l), \ldots, d_r(l))$. Because we are dealing with coadjoint orbits, $d_j(l)$ is always even; it is also easy to check that $d_j(l) - d_{j-1}(l) = 0$ or 2 for all j. (We define $d_0(l) = 0$.) Similarly, define $d_j(f) = \dim_{\mathbf{R}} H_j \cdot p_j^{\mathbf{R}}(f)$ for $f \in \mathfrak{g}_{\mathbf{R}}^*$, and set $d(f) = (d_1(f), \ldots, d_{2r}(f))$. Let $\mathcal{D} = \{d \in \mathbf{Z}^r: \exists l \in \mathfrak{g}^* \text{ with } d(l) = d\}$, and define $\mathcal{D}^{\mathbf{R}} \subseteq \mathbf{Z}^{2r}$ similarly. Define the index contraction map $\alpha: \mathcal{D}^{\mathbf{R}} \to \mathcal{D}$ by

$$\alpha(d)_j = \tfrac{1}{2}d_{2j}, \qquad 1 \le j \le r,\ d \in \mathcal{D}^{\mathbf{R}}.$$

Just as in the proof of Lemma 4.7, it is easy to see that α is a bijection.

Now define the $\mathcal{D}$-layers

$$U_d = \{l: d(l) = d\}, \qquad d \in \mathcal{D},$$

and

$$U_d^{\mathbf{R}} = \{f: d(f) = d\}, \qquad d \in \mathcal{D}^{\mathbf{R}}.$$

From (5.2), we get

$$J(U_d^{\mathbf{R}}) = U_{\alpha(d)}, \quad \text{all } d \in \mathcal{D}^{\mathbf{R}}.$$

Now define $S(d) = \{j : d_j = 2 + d_{j-1}\}$, $T(d) = \{j : d_j = d_{j-1}\}$ for d in $\mathcal{D}$ or $\mathcal{D}^{\mathbf{R}}$. It is clear that for $d \in \mathcal{D}^{\mathbf{R}}$,

$$S(\alpha(d)) = \{j_1, \ldots, j_k\} \Leftrightarrow S(d) = \{2j_1 - 1, 2j_1, \ldots, 2j_k - 1, 2j_k\}.$$

As with the $\mathcal{E}$-layers, we can order the indices in $\mathcal{D}^{\mathbf{R}}$ so that $\bigcup\{U_{d'}^{\mathbf{R}} : d' \geq d\}$ is Zariski-open in $\mathfrak{g}_{\mathbf{R}}^*$ for all $d \in \mathcal{D}^{\mathbf{R}}$; the ordering algorithm is explicit, but not unique. There is a similar ordering for $\mathcal{D}$, and we can choose the orderings so that α is order-preserving. We should note one dissimilarity between the $\mathcal{D}$-layers and the $\mathcal{E}$-layers: the U_d are *not* G-invariant in general.

Now let d be the largest index in $\mathcal{D}^{\mathbf{R}}$ such that $U_d^{\mathbf{R}}$ meets $f' + \mathfrak{k}_{\mathbf{R}}^{\perp}$. The intersection is Zariski-open in $f' + \mathfrak{k}_{\mathbf{R}}^{\perp}$. Since $J(f' + \mathfrak{k}_{\mathbf{R}}^{\perp}) = \varphi' + \mathfrak{k}^{\perp}$ and $J(U_d^{\mathbf{R}}) = U_{\alpha(d)}$, we see that $U_{\alpha(d)}$ meets $\varphi' + \mathfrak{k}^{\perp}$ in a Zariski-open set. Hence $\alpha(d)$ is the largest index in $\mathcal{E}$ such that $U_{\alpha(d)}$ meets $\varphi' + \mathfrak{k}^{\perp}$. The subspace $E_1^{\mathbf{R}}$ is defined as

$$E_1^{\mathbf{R}} = \mathbf{R}\text{-span}\{Y_j^* : j \in T(d)\}.$$

If we define

$$E_1 = \mathbf{C}\text{-span}\{X_j^* : j \in T(\alpha(d))\},$$

then $J(E_1^{\mathbf{R}}) = E_1$, because

$$(5.3) \qquad J(Y_{2j-1}^*) = X_j^*, \qquad J(Y_{2j}^*) = -iX_j^*, \qquad 1 \leq j \leq r.$$

Therefore—and this is the key fact we need about $E_1^{\mathbf{R}}$—

$$(5.4.) \qquad J(f' + E_1^{\mathbf{R}}) = \varphi' + E_1$$

In particular, $J(f' + E_1^{\mathbf{R}})$ is a *complex* affine subspace in $\mathfrak{g}^*$.

We now proceed with the proof of Theorem 1. Let $e \in \mathcal{E}^{\mathbf{R}}$ be the first index such that $U_e^{\mathbf{R}}$ meets $f' + \mathfrak{k}_{\mathbf{R}}^{\perp}$. From Theorem 3.12, this is also the first index such that $U_e^{\mathbf{R}}$ meets $f' + E_1^{\mathbf{R}}$; we saw above that this intersection is Zariski-open in $f' + E_1^{\mathbf{R}}$. Let $\varepsilon = \alpha(e)$; let $\pi_T, \pi_S : \mathfrak{g}^* = V_{T(e)} \oplus V_{S(e)} \to V_{T(e)}, V_{S(e)}$ be the projections, and define

$$\begin{aligned}
S_1^{\mathbf{R}} &= U_e^{\mathbf{R}} \cap (f' + E_1^{\mathbf{R}}), \quad S_1 = U_\varepsilon \cap (\varphi' + E_1), \\
\varphi_{\mathbf{R}} &= (\pi_T \circ P_{\mathbf{R}})|_{S_1^{\mathbf{R}}}, \qquad \varphi = (\pi_T \circ P)|_{S_1}, \\
\Sigma_1^{\mathbf{R}} &= \varphi_{\mathbf{R}}(S_1^{\mathbf{R}}) \subseteq V_{T(e)}, \quad \Sigma_1 = \varphi(S_1) \subseteq V_{T(\varepsilon)}.
\end{aligned}$$

Then S_1 is Zariski-open in the complex variety $\varphi' + E_1$. Let $k = \max\{\text{rank}_{\mathbf{C}}(d\varphi)_l : l \in S_1\}$, and set

$$S_2 = \{l \in S_1 : \text{rank}_{\mathbf{C}}(d\varphi)_l = k\}, \qquad \Sigma_2 = \varphi(S_2).$$

Because φ is rational nonsingular, S_2 is Zariski-open in $\varphi' + E_1$. Similarly, define $k_{\mathbf{R}}$ to be the maximal rank for $\varphi_{\mathbf{R}}$, and define $S_2^{\mathbf{R}} \subseteq S_1^{\mathbf{R}}$, $\Sigma_2^{\mathbf{R}} = \varphi_{\mathbf{R}}(S_2^{\mathbf{R}})$ analogously. Since $J(U_e^{\mathbf{R}}) = U_\varepsilon$ and $J(V_{T(e)}) = V_{T(\varepsilon)}$, we get

$$J\left(\Sigma_1^{\mathbf{R}}\right) = \Sigma_1, \qquad J(S_1^{\mathbf{R}}) = S_1.$$

Next, define the sets

$$W_j = P(S_j) \subseteq \Sigma_j \times V_{S(\varepsilon)}, \qquad W_j^{\mathbf{R}} = P_{\mathbf{R}}(S_j^{\mathbf{R}}) \subseteq \Sigma_j^{\mathbf{R}} \times V_{S(e)}.$$

These are just parametric versions of S_j, $S_j^{\mathbf{R}}$.

We may assume for Theorem 1 that we are in the finite multiplicity case, where $\tau_0 = 0$. In [2], we showed that $\tau_0 = 0 \Leftrightarrow k_{\mathbf{R}} = \dim_{\mathbf{R}} E_1^{\mathbf{R}}$; thus $\varphi_{\mathbf{R}} : S_2^{\mathbf{R}} \to V_{T(e)}$ is a local diffeomorphism, and for $f \in S_2^{\mathbf{R}}$ the orbit $G \cdot f$ meets $S_2^{\mathbf{R}}$ in isolated points. Moreover, $J^{-1} \circ P \circ J = P_{\mathbf{R}}$; from this it follows that $J^{-1} \circ \varphi \circ J = \varphi_{\mathbf{R}}$, so that

$$\mathrm{Rank}_{\mathbf{R}}((d\varphi)_{J(f)}) = \mathrm{Rank}_{\mathbf{R}}((d\varphi_{\mathbf{R}})_f).$$

Since the real rank of a complex analytic map is twice the complex rank, we see that $k_{\mathbf{R}} = 2k$ and that $S_2 = J(S_2^{\mathbf{R}})$.

If A is a set in $\mathfrak{g}^*$, we write $A^\#$ for its Zariski closure. We now establish the following facts:

(i) We have $W_1^\# = W_2^\#$ $(= W^\#$, by definition), $\Sigma_1^\# = \Sigma_2^\#$ $(= \Sigma^\#)$, and these sets are irreducible varieties over $\mathbf{C}$;

(ii) $\dim_{\mathbf{C}} W^\# = \dim_{\mathbf{C}} \Sigma^\# = k$.

Statement (i) is an immediate consequence of Corollary 3.8, since $P: S_k \to W_k$ and $\varphi: S_j \to \Sigma_j$ are rational nonsingular. The equality of closures holds because S_2 is Zariski-dense in S_1 and P, φ are Zariski-continuous on S_1 (Lemma 3.4).

To prove (ii), note first that

$$(5.5) \qquad W_2 \text{ contains a Zariski-open dense subset } A_1 \text{ of } W_1^\#.$$

[This is an application of Lemma 3.9, with $S = U_\varepsilon$, $T = \Sigma_\varepsilon \times V_{S(\varepsilon)}$, $\varphi = P_\varepsilon$, and $E = \varphi' + E_1$. The lemma then shows that $W_1 = P_\varepsilon(S_1)$ satisfies (5.5). But $P_\varepsilon|_{S_1}$ is birational nonsingular; Corollary 3.7 implies (5.5) for W_2.] Since $\mathrm{rank}_{\mathbf{C}}(d\varphi)_l = k = \dim_{\mathbf{C}} E_1$ if $l \in S_2$, we get

$$(5.6) \qquad \mathrm{rank}_{\mathbf{C}}(dP)_l = k, \quad \text{all } l \in S_2,$$

since $\varphi = \pi_{T(\varepsilon)} \circ P$. Consequently W_2 is covered by countably many complex manifolds in $V_{T(\varepsilon)} \times V_{S(\varepsilon)}$ having (complex) dimension k. Take any complex stratification S of $W^\#$ compatible with the set $A_1 \subseteq W^\#$ of (5.5). By definition, $\dim_{\mathbf{C}} W^\# = \max\{\dim_{\mathbf{C}} S_i : S_i \in S\}$. From (3.2), the dimension of the pieces in $W^\# \backslash A_1$ is less than the maximum dimension of the pieces in $W^\#$. Since A_1 is covered by k-dimensional submanifolds, we have $\dim_{\mathbf{C}} W^\# = \dim_{\mathbf{C}} A_1 \leq k$. On the other hand, $\dim_{\mathbf{C}} W^\# \geq \dim_{\mathbf{C}} P(S_2) = k$. Thus $\dim_{\mathbf{C}} W^\# = k$. The proof for $\Sigma^\#$ is essentially the same.

Note that $\pi_T(W^\#)$ is a Zariski dense subset of $\Sigma^\#$, since $\pi_T(W_j) = \Sigma_j$.

We are now ready to apply the main result we need from algebraic geometry. Since it applies only over $\mathbf{C}$, we have had to introduce the complex picture of the action of G on $\mathfrak{g}_{\mathbf{R}}^*$. The result may be regarded as an extension of the theorem that all polynomials in $\mathbf{C}[Z]$ of the same degree have the same number of roots (counting multiplicities); more generally, it is an extension of Bezout's theorem (see, e.g., [9]).

(5.7) THEOREM. *Let X, Y be irreducible complex varieties in $\mathbf{C}^m, \mathbf{C}^n$ respectively, of the same complex dimension, and let $p: \mathbf{C}^m \to \mathbf{C}^n$ be a polynomial map such that $p(X)$ is dense in Y. Then there is a Zariski-open nonempty set $A \subseteq Y$ such that p is a covering map from $B = p^{-1}(A) \cap X$ onto A (in the Euclidean topology). In particular, A and B are topologically connected.*

The proof is a modification of the result on pp. 117–120 of [3]; we are indebted to David Rohrlich for valuable help. From Theorem 2.25 of [3], we know that there

are normal varieties X_n, Y_n, plus morphisms (polynomial maps) μ, ν, p_n, such that
   (a) the diagram

$$
\begin{array}{ccc}
X_n & \xrightarrow{p_n} & Y_n \\
\mu\downarrow & & \downarrow\nu \\
X & \xrightarrow{p} & Y
\end{array}
$$

commutes;
   (b) there is a proper subvariety S_0 of X such that $\mu: X_n\backslash\mu^{-1}(S_0) \to X\backslash S_0$ is an isomorphism, and there is a proper subvariety S_1 of Y such that ν is an isomorphism on $Y_n\backslash\nu^{-1}(S_1)$.
The result of [3] cited above says that there is a proper subvariety S_2 of Y_n such that p_n is a covering map from $X_n\backslash p_n^{-1}(S_2)$ to $Y_n\backslash S_2$. Now let S be the Zariski closure of $S_1 \cup p(S_0) \cup \nu(S_2)$ in Y. Then $Y\backslash S$ is a nonempty Zariski-open set in Y, since $\dim S < \dim Y$ (see p. 93 of [3]), and $\nu^{-1}|_{Y\backslash S}, \mu^{-1}|_{X\backslash p^{-1}(S)}$ are isomorphisms. Since p_n is a covering map on $\mu^{-1}(X_n\backslash p^{-1}(S)) = p_n^{-1}(Y_n\backslash\nu^{-1}(S))$, we see that p is a covering map on $X\backslash p^{-1}(S)$. This proves the theorem. $\square$

   Taking $X = W^{\#}$, $Y = \Sigma^{\#}$, and $p = \pi_T$, we get a nonempty Zariski-open subset $A \subseteq \Sigma^{\#}$ such that the covering index

$$
i = i(l) = \text{card}\{\pi_T^{-1}(l) \cap W^{\#}\} = \text{Card}\{(\{l\} \times V_S) \cap W^{\#}\}
$$

is constant on A. This is not quite what we need, since for $f \in \Sigma_e^{\mathbf{R}}$ the number $n(f)$ in Theorem 3.12 is given by

$$
\begin{aligned}
n(f) &= \text{Card}((E_1^{\mathbf{R}} + f') \cap G \cdot f) = \text{Card}((E_1 + \varphi') \cap G \cdot l) \\
(5.8) \qquad &= \text{Card}(S_1 \cap P^{-1}(\{l\} \times V_S)) = \text{Card}(P(S_1) \cap (\{l\} \times V_S)) \\
&= \text{Card}(W_1 \cap (\{l\} \times V_S)), \qquad l = J(f),
\end{aligned}
$$

and thus we need to get from $W^{\#}$ to W_1. This is not hard. Let A_1 be a Zariski-open (dense) subset of $W^{\#}$ contained in W_1, as in (5.5). Then $X = W^{\#}\backslash A_1$ has complex dimension less than that of $W^{\#}$, by (3.2). Hence $\pi_T(X)$ is a countable union of manifolds in V_T having complex dimension $\leq \dim_{\mathbf{C}} X < k$. Thus $A_0 = A\backslash\pi_T(X)$ and $B_0 = \pi_T^{-1}(A_0) \cap W^{\#}$ are sets in $\Sigma^{\#}, W^{\#}$ respectively such that
   (i) $B_0 \subseteq W_1$;
   (ii) $i(l) = i$ is constant on A_0, and $\dim(\Sigma^{\#}\backslash A_0) < \dim \Sigma^{\#}$.
   From (i) and (5.8), we see that $n(f) = i$ if $f \in J^{-1}(A_0)$. Furthermore, $\dim \Sigma_2 = \dim W_2 = \dim W^{\#} = \dim \Sigma^{\#}$. Thus (ii) implies that $J^{-1}(A_0)$ has full measure in $J^{-1}(\Sigma^{\#}) = \Sigma_{\mathbf{R}}^{\#}$. Theorem 3.12 now gives

$$
\rho \cong \int_{\Sigma_1^0}^{\oplus} i\pi_f \, d\nu_1(f),
$$

which completes the proof of Theorem 1.

   6. This section is devoted to some algebraic preliminaries to the proof of Theorem 2. We write $\mathfrak{g}_0, \mathfrak{k}_0$ for the (real) Lie algebras of G_0, K_0 respectively, and $\mathfrak{g}, \mathfrak{k}$ for their complexifications; we regard $\mathfrak{k}_0, \mathfrak{g}_0$ as (real) subalgebras of $\mathfrak{k}, \mathfrak{g}$ respectively. Let G, K be the (complex) Lie groups corresponding to $\mathfrak{g}, \mathfrak{k}$ respectively. As in §4, we imbed $\mathfrak{g}_0^*$ in $\mathfrak{g}^*$, letting

$$
(If)(X + iY) = f(X) + if(Y), \quad \text{all } f \in \mathfrak{g}_0^*.
$$

We shall often write f for If. Then $\mathfrak{g}^* = \mathfrak{g}_0^* + i\mathfrak{g}_0^*$; we regard $\mathfrak{g}_0^* = \{l \in \mathfrak{g}^* : l(\mathfrak{g}_0) \subseteq \mathbf{R}\}$ as the set of real points in $\mathfrak{g}^*$. Fix a (real) Malcev basis $X_1, \ldots, X_n$ for $\mathfrak{g}_0$ and let $\{l_1, \ldots, l_n\}$ be the dual basis in $\mathfrak{g}_0^*$. These are also dual bases (over $\mathbf{C}$) for $\mathfrak{g}, \mathfrak{g}^*$, and $\mathbf{C}$-span$\{l_{j+1}, \ldots, l_n\}$ is G-invariant for all j. Define subalgebras and canonical projections,

$$\mathfrak{g}_{j,0} = \mathbf{R}\text{-span}\{X_1, \ldots, X_j\}, \qquad p_{j,0}: \mathfrak{g}_0^* \to \mathfrak{g}_{j,0}^*,$$
$$\mathfrak{g}_j = \mathbf{C}\text{-span}\{X_1, \ldots, X_j\}, \qquad p_j: \mathfrak{g}^* \to \mathfrak{g}_j^*,$$

which commute with the obvious actions of G_0, G. Write $G_0 \cdot f, G \cdot l$ for Ad*-orbits of $f \in \mathfrak{g}_0^*$, $l \in \mathfrak{g}^*$, and define dimension indices as in §4:

$$e_j(f) = \dim_{\mathbf{R}} p_{j,0}(G_0 \cdot f), \quad e(f) = (e_1(f), \ldots, e_n(f)) \ (f \in \mathfrak{g}_0^*),$$
$$\varepsilon_j(l) = \dim_{\mathbf{C}} p_j(G \cdot l), \qquad \varepsilon(l) = (\varepsilon_1(l), \ldots, \varepsilon_n(l)) \ (l \in \mathfrak{g}^*),$$

index sets $\mathcal{E}_0 = \{e \in \mathbf{Z}^n : \exists f \in \mathfrak{g}_0^* \text{ with } e = e(f)\}$, $\mathcal{E} = \{\varepsilon \in \mathbf{Z}^n : \exists l \in \mathfrak{g}^* \text{ with } \varepsilon = \varepsilon(l)\}$, and layers

$$U_e^0 = \{f \in \mathfrak{g}_0^* : e(f) = e\}, \quad e \in \mathcal{E}_0, \qquad U_\varepsilon = \{l \in \mathfrak{g}^* : \varepsilon(l) = \varepsilon\}, \quad \varepsilon \in \mathcal{E}.$$

The parametrization of Theorem 2.1 applies both to the action of G_0 on $\mathfrak{g}_0^*$ and to that of G on $\mathfrak{g}^*$. For each layer, we have cross-sectioning vector subspaces $V_{T(e)}^0, V_{T(\varepsilon)}$, and sets of orbit representatives $\Sigma_e^0, \Sigma_\varepsilon$. The connections between these objects are given by

(6.1) PROPOSITION. *In the preceding situation, we have*
(i) *$f \in \mathfrak{g}_0^* \Rightarrow e(f) = \varepsilon(If)$. Thus $\mathcal{E}^0 \subseteq \mathcal{E}$ in $\mathbf{Z}^n$.*
(ii) *For any $e \in \mathcal{E}^0$,*

$$U_e^0 = U_e \cap \mathfrak{g}_0^*, \qquad \Sigma_e^0 = \Sigma_e \cap \mathfrak{g}_0^*.$$

(iii) *One can choose orderings of $\mathcal{E}^0, \mathcal{E}$ as in Theorem 2.1(ii) so that the inclusion $\mathcal{E}^0 \subseteq \mathcal{E}$ is order-preserving.*
(iv) *For $e \in \mathcal{E}^0$, form the birational parametrizing maps*

$$P_e^0: U_e^0 \to \Sigma_e^0 \times V_{S(e)}^0, \qquad P_e: U_e \to \Sigma_e \times V_{S(e)}$$

of Theorem 2.1(iv). Then

$$V_{S(e)} = V_{S(e)}^0 \otimes_{\mathbf{R}} \mathbf{C} = V_{S(e)}^0 + iV_{S(e)}^0, \qquad V_{T(e)} = V_{T(e)}^0 + iV_{T(e)}^0$$

and P_0^e is the restriction of P_e.

PROOF. For $l \in \mathfrak{g}^*$, let $\mathfrak{h}_j = \mathfrak{h}_j(l) = \{X \in \mathfrak{g} : (\text{ad}^* X)p_j(l) = 0\}$, and set $\mathfrak{h}_0 = \mathfrak{g}$; for $f \in \mathfrak{g}_0^*$, set $\mathfrak{h}_{j,0} = \mathfrak{h}_{j,0}(f) = \{X \in \mathfrak{g}_0 : \text{ad}^*(X)p_{j,0}(f) = 0\}$, $\mathfrak{h}_{0,0} = \mathfrak{g}_0$. It follows from the proof of Theorem 2.1 that for $f \in \mathfrak{g}_0^*$,

$$e_j(f) = \text{codim}_{\mathbf{R}} \mathfrak{h}_{j,0}(f)$$

while for $l \in \mathfrak{g}^*$,

$$\varepsilon_j(l) = \text{codim}_{\mathbf{C}} \mathfrak{h}_j(l).$$

If $f \in \mathfrak{g}_0^* \subseteq \mathfrak{g}^*$, it is easy to check that $\mathfrak{h}_j(f) = \mathfrak{h}_{j,0}(f) + i\mathfrak{h}_{j,0}(f)$. This proves (i), and the first part of (ii) follows immediately. The ordering algorithm (as in §2 of [2]) implies (iii) easily. For $\varepsilon \in \mathcal{E}$, we define $S(\varepsilon) = \{j : \varepsilon_j > \varepsilon_{j-1}\}$ and

$T(\varepsilon) = \{j: \varepsilon_j = \varepsilon_{j-1}\}$; if $e \in \mathcal{E}^0 \subseteq \mathcal{E}$, these sets are the same whether we regard e as in $\mathcal{E}^0$ or $\mathcal{E}$. The spaces $V_{S(e)}, V_{T(e)} \subseteq \mathfrak{g}^*$ are the complex spans of the l_i with the appropriate indices, and $V_{S(e)}^0, V_{T(e)}^0$ are the corresponding real spaces. Obviously we have

$$V_{S(e)}^0 = V_{S(e)} \cap \mathfrak{g}_0^*, \qquad V_{T(e)}^0 = V_{T(e)} \cap \mathfrak{g}_0^* \quad \text{for } e \in \mathcal{E}^0;$$

since $\Sigma_e = V_{S(e)} \cap U_e$ and $\Sigma_e^0 = V_{S(e)}^0 \cap U_e^0$, the second part of (ii) holds.

We have now proved everything but the last claim in (iv). To construct the inverse map $Q_e = P_e^{-1}: \Sigma_e \times V_{S(e)} \to U_e$ for $e \in \mathcal{E}$, one chooses rationally varying elements $Y_j(l) \in \mathfrak{h}_{j-1}(l) \backslash \mathfrak{h}_j(l)$ such that $p_j(\text{ad}^* Y_j(l)) = l_j$ for each index $j \in S(e)$. When $l \in \mathfrak{g}_0^*$, the same construction, *verbatim*, yields $Q_e^0 = (P_e^0)^{-1}$. Since we can choose $Y_j(l) \in \mathfrak{g}_0$ when $l \in \mathfrak{g}_0^*$, the polynomials P_e, Q_e have real coefficients when $e \in \mathcal{E}_0$, and the constructions show that P_e^0, Q_e^0 are the restrictions of P_e, Q_e respectively. (Notice that if $e \in \mathcal{E}$, then $U_e \subseteq \mathfrak{g}^*$ meets $\mathfrak{g}_0^*$ iff $e \in \mathcal{E}_0$, by (i).)   □

REMARK. We do not know whether the inclusion $\mathcal{E}^0 \subseteq \mathcal{E}$ is proper in general.

(6.2) PROPOSITION. *If $f \in \mathfrak{g}_0^*$, then $G \cdot f \cap \mathfrak{g}_0^* = G_0 \cdot f$.*

PROOF. We use the geometric description of the parametrizing maps, cited in Theorem 2.1(vi). Choose a Malcev basis for $\mathfrak{g}_0$ and induce layerings of $\mathfrak{g}_0^*, \mathfrak{g}^*$ as above; let e be the index with $f_0 \in U_e^* = U_e \cap \mathfrak{g}_0^*$, and let π_S, π_T be the projections for $\mathfrak{g}^* = V_{S(e)} \oplus V_{T(e)}$. π_S is a surjective polynomial diffeomorphism of $G \cdot f$ onto $V_{S(e)}$. Similarly, if π_S^0, π_T^0 are the projections for $\mathfrak{g}_0^* = V_{S(e)}^0 \oplus V_{T(e)}^0$, then π_S^0 is a surjective polynomial diffeomorphism of $G_0 \cdot f$ onto $V_{S(e)}^0 = V_{S(e)} \cap \mathfrak{g}_0^*$. In view of Proposition 6.1(iv), π_S maps $G_0 \cdot f$ (which is in $\mathfrak{g}_0^*$) onto $\pi_S^0(G_0 \cdot f) = V_S^0$. Since π_S maps any real point in $G \cdot f$ onto V_S^0, we see that $G_0 \cdot f$ accounts for all the real points in $G \cdot f$, as required.   □

7. We now prove Theorem 2. We distinguish various Zariski closures as follows: if $S \subseteq \mathfrak{g}^*$, its (complex) Zariski hull is $S^{\mathbf{C}}$, while if $S \subseteq \mathfrak{g}_0^*$, its (real) Zariski hull in $\mathfrak{g}_0^*$ is $S^{\#}$. Pick a real Malcev basis for $\mathfrak{g}_0^*$ and partition $\mathfrak{g}_0^*, \mathfrak{g}^*$ into layers as in Theorem 2.1 and Proposition 6.1. As in Theorem 1, we may assume that $\chi \in K\hat{}$ is 1-dimensional. Thus $\chi = e^{2\pi i f}$ for some $f \in \mathfrak{k}_0^*$. Then $P^{-1}(\mathcal{O}_\chi) = f' + \mathfrak{k}_0^\perp$, where $f' \in \mathfrak{g}_0^*$ is any extension of f. Let $e \in \mathcal{E}^0$ be the first index such that U_e^0 meets $f' + \mathfrak{k}_0^\perp$. If we split $\mathfrak{k}_0^\perp = E_1^0 \oplus E_2^0$ as in Theorem 3.12, then U_e^0 is also the first layer to meet $f' + E_1^0$; their intersection is Zariski-open in $f' + E_1^0$, by Theorem 2.1(ii).

Now define

$$S_1^0 = U_e^0 \cap (f' + E_1^0),$$
$$\varphi_0 = \pi_T^0 \circ P_e^0|_{S_1^0} \quad (\text{thus } \varphi_0: S_1^0 \to V_T^0),$$
$$\Sigma_1^0 = \text{range } \varphi_0 = \varphi_0(S_1^0) \subseteq \Sigma_e^0,$$
$$k = \max\{\text{rank}_{\mathbf{R}}(d\varphi_0)_f: f \in S_1^0\},$$
$$S_2^0 = \{f \in S_1^0: \text{rank}_{\mathbf{R}}(d\varphi_0)_f = k\},$$
$$\Sigma_2^0 = \varphi_0(S_2^0) \subseteq \Sigma_1^0.$$

Then S_1^0 and S_2^0 are Zariski-open in $f' + E_1^0$ (recall that φ is rational). In §5 of [2], we showed that the condition $\tau_0 = 0$ is equivalent to $k = \dim_{\mathbf{R}} E_1^0$, or to

(7.1) $\varphi_0: S_2^0 \to \Sigma_2^0$ is a local diffeomorphism.

For the rest of the proof, we shall assume (as we may) that $\tau_0 = 0$.

Let $\varepsilon \in \mathcal{E}$ be the first index such tht U_ε meets $f' + E_1$, where $E_1 = E_1^0 + iE_1^0$. Since $U_\varepsilon \cap (f' + E_1)$ is Zariski-open in $f' + E_1$ and $f' + E_1^0$ is Zariski-dense in $f' + E_1$, we see that U_ε meets $f' + E_1^0$. Hence $\varepsilon \in \mathcal{E}^0$, and it is now clear that $\varepsilon = e$ (the "first index" for $f' + E_1^0$).

Now define the parametric versions of the sets S_j^0:

$$W_j^0 = P_e^0(S_j^0), \qquad j = 1, 2,$$

where $P_e^0: U_e^0 \to \Sigma_e^0 \times V_{S(e)}^0$ is the birational nonsingular parametrizing map for the layer U_e^0. Since $\varphi_0 = \pi_T^0 \circ P_e^0|_{S_1^0}$, we have

$$\pi_T^0(W_j^0) = \Sigma_j^0, \qquad j = 1, 2.$$

From Corollary 3.8, we know that

$(\Sigma_1^0)^\# = (\Sigma_2^0)^\# \ (= \Sigma^\#, \text{ by definition})$ is irreducible in $V_{T(e)}^0$;

$(W_1^0)^\# = (W_2^0)^\# \ (= W^\#)$ is irreducible in $\mathfrak{g}_0^*$.

From Theorem 3.12, we have

$$(7.2) \qquad \rho \cong \int_{\Sigma_1^0}^\oplus n(l)\pi_l d\nu_1(l), \qquad n(l) = \operatorname{card}(G_0 \cdot l \cap (f' + E_1^0)),$$

where $[\nu_1]$ is the canonical measure class on the semialgebraic set Σ_1^0. Moreover, W_1^0 contains a Zariski-open, dense subset of $W^\#$, by Lemma 3.9, with $S = U_e^0$, $T = \Sigma_e^0 \times V_{S(e)}^0$, $\varphi(S \cap E) = W_1^0$. Hence,

$$\dim_{\mathbf{R}}(W^\# \backslash W_1^0) < \dim_{\mathbf{R}} W^\#.$$

Since W_1^0 and S_1^0 are diffeomorphic and S_1^0 is Zariski-open in the k-dimensional space $f' + E_1^0$, we have $\dim_{\mathbf{R}} W_1^0 = k$. Thus $\dim_{\mathbf{R}} W^\# = k$; Proposition 3.3 says that

$$k = \dim_{\mathbf{R}} W^\# = \dim_{\mathbf{C}}(W^\#)^{\mathbf{C}}.$$

Because $\varphi_0: S_1^0 \to \Sigma_1^0$ is a local diffeomorphism, we can reason similarly to show that

$$k = \dim_{\mathbf{R}} \Sigma^\# = \dim_{\mathbf{C}} \left(\Sigma^\#\right)^{\mathbf{C}}.$$

For convenience, we write $W^{\mathbf{C}} = (W^\#)^{\mathbf{C}}$, $\Sigma^{\mathbf{C}} = (\Sigma^\#)^{\mathbf{C}}$. Note that $W^{\mathbf{C}}, \Sigma^{\mathbf{C}}$ are irreducible because $f' + E_1^0$ is (use Lemma 3.1 and Corollary 3.8).

Now consider $\pi_T: W^{\mathbf{C}} \to \Sigma^{\mathbf{C}}$. Theorem 5.7 applies; thus there are Zariski-open, nonempty sets $A \subseteq \Sigma^{\mathbf{C}}$, $B = \pi_T^{-1}(A) \cap W^{\mathbf{C}}$ such that $\pi_T: B \to A$ is a topological covering map. The set A is topologically connected, and there is an integer i such that

$$i = i(l) = \operatorname{card}(\pi_T^{-1}(l) \cap W^{\mathbf{C}}), \quad \text{all } l \in A.$$

Let $A_0 = A \cap \Sigma^\#$. Then A_0 is nonempty (since $\Sigma^\#$ is Zariski-dense in $\Sigma^{\mathbf{C}}$); it is easy to see that A_0 is Zariski-open in $\Sigma^\#$. Proposition 3.3 gives

$$\dim_{\mathbf{R}} \left(\Sigma^\# \backslash A_0\right) < \dim_{\mathbf{R}} \Sigma^\# = k.$$

Furthermore, A_0 must meet Σ_1^0 because $\Sigma^\# = (\Sigma_1^0)^\#$; letting $A_1 = A_0 \cap \Sigma_1^0$, we have $\Sigma_1^0 \backslash A_1 \subseteq \Sigma^\# \backslash A_0$, so that

$$(7.3) \qquad \dim_{\mathbf{R}} \left(\Sigma_1^0 \backslash A_1\right) < k.$$

Hence $\nu_1|_{A_1} = \nu_1$ in the decomposition (7.2).

We now relate $i(l)$ to $n(l)$ for $l \in A_1$. Take bases in $V^0_{T(e)}, V^0_{S(e)}$ to get coordinates (u, x) in $V^0_{T(e)} \times V^0_{S(e)}$; these bases are also bases (over $\mathbf{C}$) for $V_{T(e)}, V_{S(e)}$, and they thus give complex coordinates (w, z) in $V_{T(e)} \times V_{S(e)}$. Since $W^{\mathbf{C}}$ is the Zariski-closure of a set of real points, there are polynomials $p_1, \ldots, p_N \in \mathbf{R}[X_1, \ldots, X_n]$ such that

$$W^{\mathbf{C}} = \bigcap_{j=1}^{N} \{(w, z) \in V_{T(e)} \times V_{S(e)} = \mathfrak{g}^* : p_j(w, z) = 0\}.$$

The map π_T simply maps (w, z) to $(w, 0)$. It is clear that $(w, z) \in W^{\mathbf{C}} \Leftrightarrow (\overline{w}, \overline{z}) \in W^{\mathbf{C}}$ (where bars denote complex conjugates). In particular, if u is real, then the elements $(u, z) \in W^{\mathbf{C}}$ with $z \notin V^0_{S(e)}$ come in conjugate pairs. So if

$$j(l) = \mathrm{Card}\{\pi_T^{-1}(l) \cap W^\#\}, \qquad l \in A_1,$$

then $j(l) \equiv i(l) \equiv i \bmod 2$.

To conclude the proof of Theorem 2, it suffices to show that $j(l) = n(l)$ (where $n(l)$ is as in (7.2)) for ν_1-almost all $l \in A_1$, since in that case we will have $n(l) \equiv i \bmod 2$ a.e. The proof of this is like the proof at the end of §5. Let $X = W^\# \backslash W^0_1$. Since $W^\#$ is irreducible and W^0_1 is Zariski-open in $W^\#$, $\dim_{\mathbf{R}} X < k = \dim_{\mathbf{R}} W^\#$. Hence $\dim_{\mathbf{R}} \pi^0_T(X) < k = \dim \Sigma^0_1 = \dim \Sigma^\#$, or $\nu_1(\pi^0_T(X)) = 0$. Off $\pi^0_T(X)$, however, $j(l) = n(l)$ because their definitions are essentially the same. This completes the proof. □

REMARK. We have actually proved a bit more than Theorem 2: we have shown that except for a set of lower dimension in $\Sigma^\#$, $n(l)$ has constant parity. There are cases in which $\Sigma^\#$ is appreciably larger than Σ^1_0 (see the next section); when these cases arise, the number $n(l)$ must be (a.e.) even, since $n(l)$ is then 0 on a large subset of $\Sigma^\#$.

8. Here we give two examples, related to each other and to two examples in [**2**].

Let $\mathfrak{g}$ be the 5-dimensional complex Lie algebra spanned by X_1, X_2, X_3, X_4, X_5, with nonzero brackets $[X_5, X_4] = X_3$, $[X_5, X_3] = X_2$, $[X_5, X_2] = X_1$. We denote by $\mathfrak{g}_0$ the real algebra with the same brackets.

The given basis is a Malcev basis for $\mathfrak{g}$, with dual basis $l_1, \ldots, l_5$ in $\mathfrak{g}^*$. The elements of $\mathcal{E}$ and the corresponding layerings and decompositions are:

(a) $e_1 = (0, 1, 1, 1, 2); U_{e_1} = \{l = \sum_{j=1}^{5} \alpha_j l_j : \alpha_1 \neq 0\}$, $V_S = \mathbf{C}\text{-span}\{l_2, l_5\}$, $V_T = \mathbf{C}\text{-span}\{l_1, l_3, l_4\}$, $\Sigma_{e_1} = \{\alpha_1 l_1 + \alpha_3 l_3 + \alpha_4 l_4 : \alpha_1 \neq 0\}$, and the orbit map $Q = Q_{e_1}$ is given by

$$Q(\alpha_1 l_1 + \alpha_3 l_3 + \alpha_4 l_4; z l_2 + w l_5)$$
$$= \alpha_1 l_1 + z l_2 + \left(\alpha_3 + \frac{z^2}{2\alpha_1}\right) l_3 + \left(\alpha_4 + \frac{\alpha_3 z}{\alpha_1} + \frac{z^3}{6\alpha_1^2}\right) l_4 + w l_5.$$

(b) $e_2 = (0, 0, 1, 1, 2); U_{e_2} = \{l = \sum_{j=2}^{5} \alpha_j l_j : \alpha_2 \neq 0\}$, $V_S = \mathbf{C}\text{-span}\{l_3, l_5\}$, $V_T = \mathbf{C}\text{-span}\{l_1, l_2, l_4\}$, $\Sigma_{e_2} = \{\alpha_2 l_2 + \alpha_4 l_4 : \alpha_2 \neq 0\}$, and $Q = Q_{e_2}$ is given by

$$Q(\alpha_2 l_2 + \alpha_4 l_4; z l_3 + w l_5) = \alpha_2 l_2 + z l_3 + \left(\frac{z^2}{2\alpha_2} + \alpha_4\right) l_4 + w l_5.$$

(c) $e_3 = (0,0,0,1,2); U_{e_2} = \{l = \sum_{j=3}^{5} \alpha_j l_j : \alpha_3 \neq 0\}, V_S = \text{C-span}\{l_4, l_5\}, V_T = \text{C-span}\{l_1, l_2, l_3\}, \Sigma_{e_3} = \{\alpha_3 l_3 : \alpha_3 \neq 0\}$, and $Q = Q_{e_3}$ is given by

$$Q(\alpha_3 l_3; z l_4 + w l_5) = \alpha_3 l_3 + z l_4 + w l_5.$$

(d) $e_4 = (0,0,0,0,0); U_{e_4} = \{l = \sum_{j=4}^{5} \alpha_j l_j\}, V_S = \{0\}, V_T = \mathfrak{g}^*, \Sigma_{e_4} = U_{e_4}$, and $Q: \Sigma_{e_4} \times \{0\} \to \Sigma_{e_4}$ is the obvious map.

We identify $\text{Ad}^*(G)$-orbits in $\mathfrak{g}^*$ with elements of $G^\wedge$ (where G is the simply connected group corresponding to $\mathfrak{g}$) via the Kirillov correspondence and the J-map. The representation corresponding to l in some Σ_{e_j} will be denoted by π_l.

1. Let $\mathfrak{k}_1 = \text{C-span}(X_4)$, and let $\sigma = \chi_\alpha$ correspond to the functional $\alpha(X_4) = \alpha$. Then α is the restriction to $\mathfrak{k}_1$ of αl_4, and

$$\alpha l_4 + \mathfrak{k}_1^\perp = \{\alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 + \alpha l_4 + \alpha_5 l_5 : \alpha_1, \alpha_2, \alpha_3, \alpha_5 \in \text{C}\}.$$

Hence $\alpha l_4 + \mathfrak{k}_1^\perp$ meets U_{e_1}. If $l = \alpha_1 l_1 + \alpha_3 l_3 + \alpha_4 l_4 \in \Sigma_{e_1}$, then

$$G \cdot l \cap (\alpha l_4 + \mathfrak{k}_1^\perp) = \left\{ l + z l_2 + \frac{z^2}{2\alpha_1} l_3 + \left(\frac{\alpha_3 z}{\alpha_1} + \frac{z^3}{6\alpha_1^2} \right) l_4 + w l_5 : \right.$$
$$\left. w, z \in \text{C}, \frac{z^3}{6\alpha_1^2} + \frac{a_3 z}{\alpha_1} + \alpha_4 = \alpha \right\}.$$

This set consists (generically) of three complex lines, one for each of the roots of the cubic polynomial. Each line is an $\text{Ad}^*(K_1)$-orbit. Thus

$$\text{Ind}(K_1 \uparrow G; \chi_\alpha) \cong \int_{\Sigma_{e_1}}^{\oplus} 3\pi_l \, d\nu(l),$$

where $K_1 = \exp \mathfrak{k}_1$ and $\nu(l)$ is 3-dimensional Lebesgue measure.

2. Let $\mathfrak{k}_2 = \text{C-span}(X_1, X_4)$ and let $\sigma = \chi_\beta$ correspond to the functional $\beta(X_4) = \beta$, $\beta(X_1) = 0$. Then β is the restriction to $\mathfrak{k}_2$ of βl_4, and

$$\beta l_4 + \mathfrak{k}_2^\perp = \{\alpha_2 l_2 + \alpha_3 l_3 + \beta l_4 + \alpha_5 l_5 : \alpha_2, \alpha_3, \alpha_5 \in \text{C}\}.$$

Hence $\beta l_4 + \mathfrak{k}_2^\perp$ misses U_{e_1}, but meets U_{e_2}. For $l = \alpha_2 l_2 + \alpha_4 l_4 \in \Sigma_{e_2}$,

$$\mathcal{O}_l \cap (\beta l_4 + \mathfrak{k}_2^\perp) = \left\{ l + z l_3 + \frac{z^2}{2\alpha_2} l_4 + w l_5 : z, w \in \text{C}, \frac{z^2}{2\alpha_2} + \alpha_4 = \beta \right\}$$

a union (generically) of two complex lines corresponding to the roots of the polynomial $z^2/2\alpha_2 + \alpha_4 = \beta$. Each line is an $\text{Ad}^*(K_2)$-orbit. Hence

$$\text{Ind}(K_2 \uparrow G; \chi_\beta) \cong \int_{\Sigma_{e_2}}^{\oplus} 2\pi_l \, d\nu(l),$$

where $\nu(l)$ is 2-dimensional Lebesgue measure.

These examples correspond to Examples 4 and 2 of [2], where we performed calculations equivalent to the above ones for $\mathfrak{g}_0$ and $\mathfrak{k}_{1,0}, \mathfrak{k}_{2,0}$ (these latter are the real forms of the $\mathfrak{k}_j$). For $\mathfrak{k}_{2,0}$, the multiplicity was always 2, but the integral was taken over $\{l = \alpha_2 l_2 + \alpha_4 l_4 \in \Sigma_{e_2}^0 : \alpha_2, \alpha_4 \in \text{R} \text{ and } \alpha_2(\alpha_4 - \beta) > 0\}$. The intersection of the Zariski closure of this set with $\Sigma_{e_2}^0$ is all of $\Sigma_{e_2}^0$; as the note at the end of §7 states, the parity of the multiplicity is constant on (almost) all of $\Sigma_{e_2}^0$.

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